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Layered tropical commutative algebra

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Tiivistelmä

Trooppisissa puolirenkaissa käytettävä max-plus-algebra tekee kaikista puolirenkaan alkioista idempotentteja (yhteenlaskun suhteen). Tämän ilmiön estämiseksi Izhakian ja muut ovat esitelleet kerrostetun trooppisen matematiikan [15], missä puolirenkaan alkiot on jaoteltu (erillisiin) kerroksiin siten, että summa kuuluu eri kerrokseen kuin summattavat.

Työn tarkoituksena on löytää trooppisia vastineita joillekin kommutatiivisen algebran käsitteille. Eräs keskeinen käsite kommutatiivisessa algebrassa on renkaan ideaali, mutta koska puolirenkaassa ei välttämättä ole yhteenlaskun vasta-alkioita, ideaalien sijasta on keskitytty kongruensseihin. Muita keskeisiä käsitteitä kommutatiivisessa algebrassa (kuten myös algebrallisessa geometriassa) ovat algebralliset joukot ja varistot. Kun ideaalit korvataan kongruensseilla, voidaan puhua kongruenssivaristoista. Ne koostuvat sellaisista pisteistä, joissa tiettyjen polynomiparien arvot ovat samat.

Tavanomaisessa kommutatiivisessa algebrassa varistot ja algebralliset joukot ovat (suunnilleen) sama asia. Tässä työssä on onnistuttu todistamaan vastaava yhteys trooppisten algebrallisten joukkojen ja kongruenssivaristojen välillä. Jokainen algebrallinen joukko voidaan esittää kongruenssivaristona, ja sama pätee myös toisin päin. Tämä on uusi tulos trooppisessa matematiikassa. Tuloksen saavuttamiseksi tarvitaan Izhakianin ja muiden esittämää kerrostettua trooppista algebraa rykelmäjuurineen [15] sekä Bertramin ja Eastonin esittämää kierrettyä tuloa [1].

Algebrallisten joukkojen ja kongruenssivaristojen välisen yhteyden löytyminen edellytti myös joidenkin algebrallisesta geometriasta tuttujen tulosten todistamista trooppisessa algebrassa sekä trooppisiin alueisiin liittyvän uuden teorian kehittämistä. Trooppinen alue koostuu pisteistä, missä polynomin arvo määräytyy yhden monomin perusteella, eli yhden monomin arvo on aidosti suurempi kuin muiden monomien.

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Abstract

Max-plus algebra applied in tropical semirings has the effect that all elements become additively idempotent. To prevent such a phenomenon, Izhakian et al. [15] have introduced layered tropical mathematics, where a semiring is decomposed into (disjoint) layers such that the sum of the two elements belongs to a layer, which is different from that of the summands.

The purpose of this work is to find out tropical counterparts for certain concepts of commutative algebra. One of such central concepts in commutative algebra is an ideal of a ring. However, semirings do not necessarily have additive inverses, and thus, it is more reasonable to concentrate on congruences instead of ideals. Other central concepts in commutative algebra (as well as in algebraic geometry) are algebraic sets and varieties. By replacing ideals with congruences, we can speak on congruence varieties, which means the set of points, where certain pairs of polynomials reach the same values.

In usual commutative algebra, varieties and algebraic sets are very much the same. In this work, we have succeeded to prove that there is a similar kind of correspondence between tropical algebraic sets and congruence varieties. Each algebraic set can be expressed in the terms of a congruence variety, and vice versa. This is a new result in tropical mathematics. The crucial concepts in achieving the result are layered tropical algebra with cluster roots from Izhakian et al. [15] and twisted product of congruences from Bertram and Easton [1].

Finding out the connection between tropical algebraic sets and congruence varieties demanded new tropical proofs for some results familiar from usual algebraic geometry and new theory on tropical regions. A tropical region consists of points, where a single monomial reaches a strictly greater value than the other monomials in a polynomial.

Acknowledgements

Completing this thesis took me about two years. It has been a way to keep me busy during my unemployment period. It has also made me familiar with mathematical research. Namely, during my advanced studies in mathematics, I often wondered, what mathematical research is like. Now I have some conception of it.

First of all, I thank my supervisor, professor Eero Hyry, for giving me such an interesting research topic, for putting so much interest and time for this work, for giving me opportunity for mathematical conversations, and for showing me the points, where I have not been thorough enough. I thank professor Erkki Mäkinen (in computer science) for giving references for several grant applications, and for all other kind of support.

I thank those plenty of foundations and other institutions that never admitted funding for this work. In this way, I have been able to do this work without any obligations and to weigh my motivation towards mathematical research. I can also say that I have not made this thesis for money or for merit, but for fun and for joy.

Last but not least, I thank my family for all kind of support during this work and for providing me other kind of activities different from this work. I apologize for being absent-minded so often.

Lempäälä, in autumn 2015
Maarit Harsu

Contents

Chapter 1

Introduction

This work considers tropical geometry and especially an algebraic approach to it. The word ”tropical” refers to the computer scientist Imre Simon and to his home country Brazil. So far most of the approaches to tropical geometry have been combinatorical in nature, but the purpose in this work is to find a direct algebraic connection to tropical geometry.

Basic elements in tropical geometry are semirings [11] with a certain kind of arithmetics. A real tropical semiring can be constructed as follows: take the real addition as the tropical multiplication and define the tropical addition to be the maximum between two real numbers. More generally, a tropical semiring can be based on any ordered monoid: the monoid operation becomes the tropical multiplication, while the tropical addition is defined to be the maximum between two monoid elements, when the maximum is determined based on the order of the monoid.

The above kind of aritmetics makes calculations easier (and cheaper when dealing with computers). For example, to calculate the value of a polynomial of any degree, we calculate the values of (a finite number of) polynomials of degree one, at most. However, from the algebraic point of view, the above kind of structure is rather weak. Namely, when taking the sum (i.e. maximum) between two identical elements, the result is the element itself. In other words, all elements of a tropical semiring are additively idempotent.

To deal with the above weakness, Izhakian (et al.) have proposed so called supertropical algebra, or a layered tropical algebra as a generalization of the former one. Each element of a layered tropical semiring has a layer, which is an element of another (non-tropical) semiring. When adding two identical elements, the layer of the sum is greater than those of the summands, and thus, the sum is different from either of the summands.

This thesis is based on the work authored by Izhakian (et al.). The original purpose (on August 2013) was especially to concentrate on their article on ideals [20]. To better understand it, it was necessary to check some details from other papers written by the same authors, especially from [14], [15], [17], [19]. During this task, I realized that ideals do not play so central role with semirings than they do with rings. This is because, a semiring need not contain additive inverses. Instead of ideals, it is more fruitful to work with congruences. Moreover, congruences become very ideal-like by applying a special kind of operation, twisted product, introduced by Bertram and Easton [1]. Actually, the main contribution of this thesis, i.e. the correspondence between congruence varieties and tropical algebraic sets, was achieved by combining the results presented by Izhakian (et al.) with those introduced by Bertram and Easton.

Due to the process described above, the final content of this thesis is rather far away from the visio that I had when I started. But this is the nature of research: at the beginning you do not know what the result will be, or you do not know where you finally end up.

The final structure of this thesis is as follows. The second chapter deals with basic (or real) tropical mathematics, as presented e.g. in [24], and introduces some basic algebraic concepts that are crucial in tropical mathematics. The third chapter concentrates on the basic elements of Izhakian's (et al.) work such as a layered tropical semiring, while the fourth chapter considers polynomials over them. The fifth chapter is a prepatory chapter for the next (and last) one. It introduces the concepts to be needed in the sixth chapter. One of such concepts is a congruence and the twisted product of it [1]. Another important concept is a tropical region, which is invented by myself (but inspired by Izhakian (et al.)).

The last two chapters comprise the most important (and most interesting) part of the thesis. They introduce new knowledge on the correspondence between congruence varieties and affine layered algebraic sets, analogical to the traditional relationship between varieties and affine algebraic sets. The new knowledge started to born based on [18, p. 32], where a congruence variety was put in the place of an algebraic set without any explanation. Searching for the explanation led me to the source of the new knowledge.

The reader is expected to be familiar with commutative algebra (as presented e.g. in [27] and [28]) as well as algebraic geometry (as presented e.g. in [8] and [26]). Category theory occurs in some parts of this work, but its role is very small, so it does not matter, if the reader is not familiar with it. No knowledge about tropical geometry is expected. Instead, the second chapter acts as an introduction to this subject. To keep it short enough, it may lack mathematical exactness. For example, some concepts are used without an exact definition, when their definition is postponed to subsequent chapters, or some concepts are defined in a narrow sense, when they will be defined more generally in subsequent chapters. Tropical mathematics is introduced only on those parts that is necessary for the rest of the work. A more in-depth presentations about tropical mathematics can be found e.g. in [13] and [24].

In this work, I have used terms "proposition" and "lemma" in the following way. A proposition is used, when the claim (but not necessarily the proof) is taken from some other publication. A lemma is used, when also the claim is formulated by myself. I have also nominated those claims as lemmata that occur in usual algebraic geometry, but I have not found their tropical counterparts from any publication (perhaps they do not exist), and thus, I have formulated them by myself.

As I mentioned at the beginning, the approaches to tropical geometry are mainly combinatorical. One exception is the work authored by Izhakian (et al.). However, quite recently, other works sharing algebraic approach have been published [1], [10], [22], [23], [25]. These works have not been introduced in this thesis (except for the twisted product from [1]). Instead, the comparison between them and the work of Izhakian (et al.) is the topic of the future research.

Chapter 2

Backgrounds

2.1 Real tropical semirings

This chapter first introduces real tropical semirings and polynomials over them, and shows that calculations in these structures are simpler than usual real operations. After that it presents how to move to real tropical structures from any field in order to achieve such simple calculations.

We start by introducing semirings in general.

Definition 2.1. The pair $(R, \circ)$ is called a *semigroup*, if the operation $\circ$ is associative, and if $a \circ b \in R$, for all $a, b \in R$. In other words, a semigroup is a group without the requirements of a neutral and inverse elements. If a semigroup comprises a neutral element, it is called a *monoid*.

Definition 2.2. The triple $(R, +, \cdot)$, with $+$ for addition and $\cdot$ for multiplication, is called a *semiring*, if

(i) $(R, +)$ is an Abelian monoid (0 as a typical neutral element),

(ii) $(R, \cdot)$ is a monoid (1 as a typical neutral element),

(iii) multiplication distributes over addition,

(iv) the additive neutral element annihilates R, i.e. $a \cdot 0 = 0 = 0 \cdot a$, for all $a \in R$.

If R is a semiring such that $(R, \cdot)$ is Abelian, then R is said to be *commutative*. If not otherwise mentioned, we assume semirings to be commutative.

If R is a semiring such that each non-zero element in R has a multiplicative inverse, then R is called a *semifield*. In this case, $(R\backslash\{0\}, \cdot)$ is a group.

Definition 2.3. Let $(R, +, \cdot)$ be a (commutative) semiring and $S \subset R$. It is said that S is a *subsemiring* of R, if $0, 1 \in S$ and $a + b, ab \in S$, for all $a, b \in S$.

Definition 2.4. Let R and S be semirings. The map $f : R \to S$ is called a *(semiring) homomorphism*, if

(i) $f(a+b) = f(a) + f(b)$,

(ii) $f(ab) = f(a)f(b)$,

(iii) $f(0_R) = 0_S$,

(iv) $f(1_R) = 1_S$,

for all $a, b \in R$.

Consider R and S as (multiplicative) monoids. Then f is a *monoid homomorphism*, if it satisfies properties (ii) and (iv) above.

Now we are ready to move to real tropical semirings.

Definition 2.5. Let $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ be a semiring. It is called a *real tropical semiring*, if the operations for addition and multiplication are defined as

(i) $x \oplus y = \max\{x, y\}$,

(ii) $x \odot y = x + y$,

for all $x, y \in \mathbb{R} \cup \{-\infty\}$. Due to the above kind of arithmetic, the real tropical semiring can also be called a *max-plus algebra*. Real tropical mathematics often uses the denotation $\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$.

It can also be talked about *min-plus algebra*, if the semiring in question is $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ and maximum is replaced with minimum in (i). In this case, $\mathbb{T} := (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$. However, we take max-plus algebra as default, and assume the former interpretation for $\mathbb{T}$.

Remark. In the real tropical semiring, $-\infty$ is the additive neutral element or zero (element), since

$$x \oplus -\infty = x = -\infty \oplus x,$$

for all $x \in \mathbb{T}$. Correspondingly, 0 is the multiplicative neutral element or unit (element), since

$$x \odot 0 = x = 0 \odot x,$$

for all $x \in \mathbb{T}$.

Actually, $\mathbb{T}$ is a semifield, since all non-zero elements have multiplicative inverses, i.e. it holds that

$$x \odot (-x) = x + (-x) = 0,$$

for all $x \in \mathbb{T} \setminus \{-\infty\}$.

However, $\mathbb{T}$ is not a ring, since there is no additive inverses for non-zero elements. Namely, if $a \in \mathbb{T}\backslash\{-\infty\}$, then there is no element $b \in \mathbb{T}$ to satisfy

$$a \oplus b = -\infty.$$

The operation of taking maximum requires that the set (here $\mathbb{R}$) must be totally ordered. Joining the element $-\infty$ has no effect on the order, because it holds $x > -\infty$, for all real numbers x.

Max-plus algebra leads to the following characteristics, but we give them more generally for any semiring.

Definition 2.6. Let $(R, +, \cdot)$ be a semiring.

(i) If $a \in R$ such that $a + a = a$, then a is *(additively) idempotent*. If all the elements in R are idempotent, then R itself is called *idempotent*.

(ii) If $a + b \in \{a, b\}$, for all $a, b \in R$, then R is *bipotent*.

(iii) If

$$a \cdot c = b \cdot c \quad \text{implies} \quad a = b,$$

for all $a, b \in R$ and $0 \neq c \in R$, then R is *(multiplicatively) cancellative*. More generally, an algebraic structure with a single operation is *cancellative*, if the above implication holds true for the operation.

Remark. Bipotency implies idempotency.

Example 2.7. The real tropical semiring $\mathbb{T}$ is both idempotent and bipotent. Moreover, multiplication in $\mathbb{T}$ is cancellative. This is due to the fact that the normal addition between real numbers is cancellative. Instead, addition in $\mathbb{T}$ is not cancellative, since $a \oplus c = b \oplus c$ does not imply $a = b$. Namely, subtraction is not defined due to the lack of the additive inverses.

In ring theory, an integral domain is defined to be a non-zero ring with no zero divisors. Due to the lack of additive inverses, the corresponding concept in semirings is described with the cancellative property of multiplication. The following proposition shows the equivalence between these descriptions.

Proposition 2.8. *Let $0 \neq R$ be a commutative ring. Multiplication in R is cancellative, if and only if R has no zero divisors.*

Proof. Let $a, b, c \in R$ such that $c \neq 0$. It is required to show that

$$ac = bc \text{ implies } a = b \qquad \Longleftrightarrow \qquad ab = 0 \text{ implies } a = 0 \text{ or } b = 0.$$

"$\Rightarrow$" Suppose that $ab = 0$ and $b \neq 0$. Now $ab = 0 = 0 \cdot b$, when the assumption implies $a = 0$. Hence, $a = 0$ or $b = 0$.

"$\Leftarrow$" Suppose that $ac = bc$. Since R is a ring with additive inverses, we can write this equation in the form $(a - b)c = 0$. Since $c \neq 0$, the assumption implies $a - b = 0$, and thus, $a = b$, due to the additive inverses. $\square$

The following definition introduces a structure corresponding to an integral domain in ring theory.

Definition 2.9. If a non-zero semiring is commutative and cancellative, it is called a *semidomain*.

Max-plus algebra (with total order) leads to many simplifications in calculations. For instance, real tropical semirings follow so called Frobenius property:

Proposition 2.10. *Let* $m \in \mathbb{N}$*. Then*

$$(\bigoplus_{i=1}^{n} a_i)^m = \bigoplus_{i=1}^{n} a_i^m,$$

for all $a_i \in \mathbb{T}$ *(*$i \in \{1, \dots, n\}$*).*

Proof. Suppose that $a_j = \max\{a_1, \dots, a_n\}, (1 \leq j \leq n)$. Then

$$\begin{aligned}
(\bigoplus_{i=1}^{n} a_i)^m &= (a_1 \oplus \cdots \oplus a_n)^m \\
&= (\max\{a_1, \dots, a_n\})^m \\
&= a_j^m \\
&= \max\{a_1^m, \dots, a_n^m\} \\
&= a_1^m \oplus \cdots \oplus a_n^m \\
&= \bigoplus_{i=1}^{n} a_i^m.
\end{aligned}$$

□

Remark. A special case of the above property is called the Freshman's dream, meaning that

$$(a \oplus b)^m = a^m \oplus b^m,$$

for all $a, b \in \mathbb{T}$ and $m \in \mathbb{N}$.

2.2 Real tropical polynomials

Since multiplication is the normal real addition in max-plus algebra, exponentiation means multiplication between the base and the power. More precisely,

$$x^n = \underbrace{x \odot \cdots \odot x}_{n \text{ pieces}} = \underbrace{x + \cdots + x}_{n \text{ pieces}} = nx,$$

for all $x \in \mathbb{T}$ and $n \in \mathbb{N}$. In tropical literature, x^n is sometimes written as $x^{\odot n}$ to emphasize tropical exponentiation.

We will mainly allow only $\mathbb{N}$ as the set of exponents in polynomials, although in tropical literature, the sets $\mathbb{Z}$ and $\mathbb{Q}$ can be possible. However, we will occasionally speculate the case of $\mathbb{Z}$ or move temporarily to it.

Definition 2.11. A *real tropical polynomial* in n indeterminants is defined as

$$F = \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} a_{\boldsymbol{i}} X_1^{i_1} \odot \cdots \odot X_n^{i_n},$$

where $a_{\boldsymbol{i}} \in \mathbb{T}$ for all $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ such that $a_{\boldsymbol{i}} \neq -\infty$ only with finitely many $\boldsymbol{i} \in \mathbb{N}^n$. For short, we can write

$$F = \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} a_{\boldsymbol{i}} \boldsymbol{X}^{\boldsymbol{i}},$$

where $\boldsymbol{X}^{\boldsymbol{i}}$ stands for $X_1^{i_1} \odot \cdots \odot X_n^{i_n}$.

Each term $a_{\boldsymbol{i}} \boldsymbol{X}^{\boldsymbol{i}}$ is called a *monomial* of F. The set of real tropical polynomials in n indeterminants is denoted as $\mathbb{T}[X_1, \ldots, X_n]$.

As in real arithmetics, we can drop the tropical multiplication sign away, if such an action does not cause confusion. Therefore, multiplication signs are not typically written in polynomials, as is the case in the following example.

Example 2.12. Consider the general form of a real tropical polynomial of degree three,

$$F = aX^3 \oplus bX^2 \oplus cX \oplus d \in \mathbb{T}[X].$$

When interpreting the tropical operations as given in Definition 2.5, we obtain

$$F = \max\{3X + a, 2X + b, X + c, d\}.$$

Now each monomial corresponds to a line. To calculate the points, where each monomial reaches greater values than the other monomials, we assume that

$$d - c \leq c - b \leq b - a. \tag{2.1}$$

For example, aX^3 gives the maximum at the points, for which it holds

$$3x + a \geq 2x + b \quad \text{and} \quad 3x + a \geq x + c \quad \text{and} \quad 3x + a \geq d. \tag{2.2}$$

The first inequation can be written as $x \geq b - a$, and the second one implies

$$2x \geq c - a = c - b + b - a \geq 2(c - b),$$

when recalling the inequations in (2.1). Therefore $x \geq c - b$. Similarly, the third inequation in (2.2) implies

$$3x \geq d - a = d - c + c - b + b - a \geq 3(d - c).$$

As a conclusion, all three inequations in (2.2) hold true at the points $x \geq b - a$. By performing similar kind of calculations for each monomial, we end up to the piecewise linear function

$$f : \mathbb{T} \to \mathbb{T}, f(x) = \begin{cases} 3x + a, & \text{if } b - a \leq x \\ 2x + b, & \text{if } c - b \leq x \leq b - a \\ x + c, & \text{if } d - c \leq x \leq c - b \\ d, & \text{if } x \leq d - c. \end{cases}$$

The graph of the function is depicted in Figure 2.1.

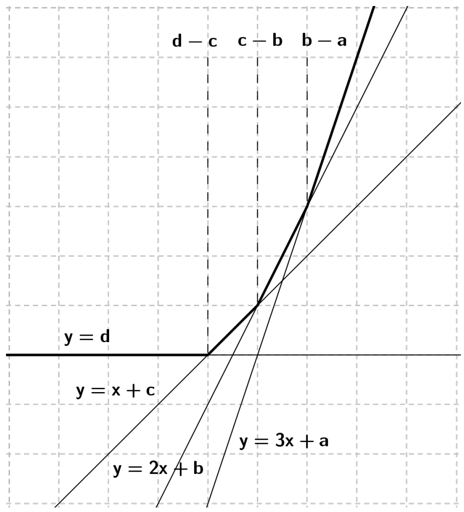


Figure 2.1: The graph of the function determined by the polynomial $F = aX^3 \oplus bX^2 \oplus cX \oplus d$ (drawn in bold line segments and rays).

As can be seen (and easily concluded), the graph determined by a real tropical polynomial in one indeterminate is always piecewise linear. Figure 2.1 depicts also the points, where the function is non-differentiable, i.e. the points, where the graph bends. These points are the cutting points of the four lines corresponding the monomials in the original polynomial.

The inequations in (2.1) in Example 2.12 have the effect that each line in Figure 2.1 contribute. This is not always the case, and thus, we give the following definition.

Definition 2.13. Let $a \in \mathbb{T}^n$, and $F \in \mathbb{T}[X_1, \ldots, X_n]$, and write it as a sum of monomials, $F = \sum_{i=1}^{r} F_i$. Let $j \in \{1, \ldots, r\}$, and denote

$$H_j := \sum_{i=1, i \neq j}^{r} F_i,$$

the polynomial without a certain monomial F_j. (If F is a monomial, then $H_j = -\infty$.) A monomial F_j is *essential (in F) at a*, if $F_j(a) > H_j(a)$. A monomial F_j is *essential (in F)*, if it is essential at some $a \in \mathbb{T}^n$.

Remark. If F_j is not essential, then $F_j(a) \leq H_j(a)$, for all $a \in \mathbb{T}^n$. This implies $H_j(a) = F(a)$, for all $a \in \mathbb{T}^n$. Namely, now F_j does not affect on the value of F. (Definition 4.15 will express a more general counterpart for this concept.)

Example 2.14. Consider the polynomial $F = X^3 \oplus X^2 \oplus 4X \oplus 3 \in \mathbb{T}[X]$. The monomial X^2 is not essential in F. Namely, if was essential, then there would exist $x \in \mathbb{T}$ such that

$$2x > 3x \quad \text{and} \quad 2x > x + 4 \quad \text{and} \quad 2x > 3,$$

but the first and the second inequations cannot hold at the same time. All the other monomials in F are essential.

Since X^2 is not essential in F, the polynomials $X^3 \oplus X^2 \oplus 4X \oplus 3$ and $X^3 \oplus 4X \oplus 3$ represent the same function:

$$f : \mathbb{T} \to \mathbb{T}, f(x) = \begin{cases} 3x, & \text{if } 2 \leq x \\ x + 4, & \text{if } -1 \leq x \leq 2 \\ 3, & \text{if } x \leq -1. \end{cases}$$

The situation is depicted in Figure 2.2.

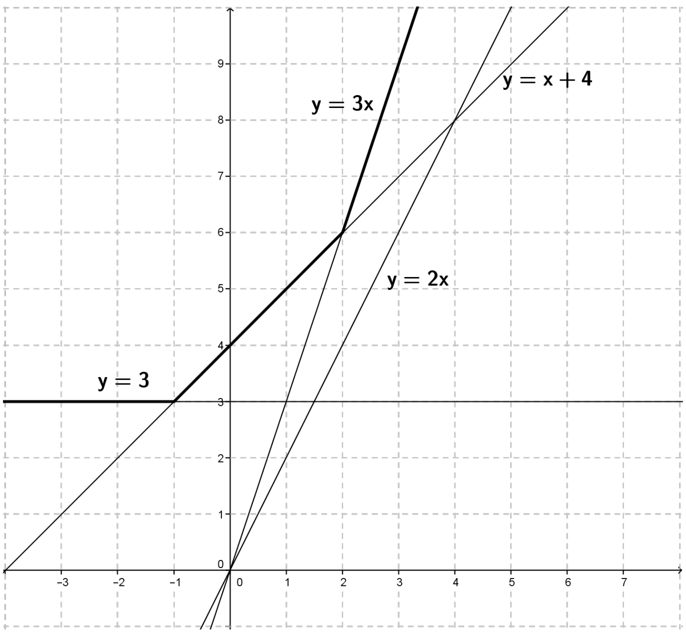


Figure 2.2: The graph of the function determined by the polynomial $F = X^3 \oplus X^2 \oplus 4X \oplus 3$ (drawn in bold line segments and rays).

If we consider polynomial functions instead of polynomials, the fundamental theorem of algebra holds tropically in the case of one indeterminant

[12, p. 6]. It says that every real tropical polynomial function with rational coefficients can be written uniquely as a product of tropical linear functions. For instance, in the case of Example 2.12 and Figure 2.1, we have

$$
\begin{aligned}
F &= aX^3 \oplus bX^2 \oplus cX \oplus d \\
&= a(X^3 \oplus (b-a)X^2 \oplus (c-a)X \oplus (d-a)) \\
&= a(X^3 \oplus (b-a)X^2 \oplus (b-a)(c-b)X \oplus (d-a)) \\
&= a(X^3 \oplus (b-a)X^2 \oplus (c-b)X^2 \oplus (d-c)X^2 \\
&\qquad\quad \oplus (b-a)(c-b)X \oplus (b-a)(d-c)X \oplus (c-b)(d-c)X \\
&\qquad\quad \oplus (b-a)(c-b)(d-c)) \\
&= a(X \oplus (b-a))(X \oplus (c-b))(X \oplus (d-c)).
\end{aligned}
$$

Note that this factorization requires that we select such a representative polynomial for the polynomial function, all the monomials of which are essential.

2.3 Roots of real tropical polynomials

The factorization presented just previously leads to the roots of polynomials. In tropical mathematics, zero points cannot be used in defining roots, because the zero element $-\infty$ is achieved only by a zero polynomial or by a polynomial with no constant term evaluated at $-\infty$. Therefore it is reasonable to define roots in the following way.

Definition 2.15. Let

$$F = \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} a_{\boldsymbol{i}} X_1^{i_1} \odot \cdots \odot X_n^{i_n} \in \mathbb{T}[X_1, \ldots, X_n]$$

be a real tropical polynomial. It is said that $x \in \mathbb{T}^n$ is a *root of F*, if there exist $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^n$ such that $\boldsymbol{i} \neq \boldsymbol{j}$ and

$$F(x) = (a_{\boldsymbol{i}} X^{\boldsymbol{i}})(x) = (a_{\boldsymbol{j}} X^{\boldsymbol{j}})(x).$$

Remark. In other words, a root is a point, where at least two monomials in the polynomial reach the maximum value, and thus, this value becomes the value of the whole polynomial. At such points, the graph of the piecewise linear function bends and creates a corner. Therefore they are also called *corner roots*. The set of corner roots is called a *corner locus*. We denote the corner locus of F as $\mathscr{Z}_{\mathrm{corn}}(F)$. (Definition 4.21 will express a more general counterpart for this concept.)

The above definition is equivalent to that given e.g. in [13, p. 11] (Definition 1.11). Besides the points, where at least two monomials reach the maximum value, also the points, where the polynomial evaluates to zero ($-\infty$), can be counted in corner roots, as has been done in [4, p. 4] (Definition 2.1).

Example 2.16. Consider again the polynomial $F = X^3 \oplus 4X \oplus 3 \in \mathbb{T}[X]$. Now,

$$F(-1) = -3 \oplus 3 \oplus 3 = 3,$$
$$F(2) = 6 \oplus 6 \oplus 3 = 6,$$

which means that the above points are roots of F, since at both of these points, two monomials give the maximum value. Figure 2.2 shows that these points are also the cutting points of the lines representing the monomials.

Example 2.17. Consider the polynomial $F = X^3 \oplus 3X^2 \oplus 4X \oplus 3 \in \mathbb{T}[X]$. Now,

$$F(-1) = -3 \oplus 1 \oplus 3 \oplus 3 = 3,$$
$$F(1) = 3 \oplus 5 \oplus 5 \oplus 3 = 5,$$
$$F(3) = 9 \oplus 9 \oplus 7 \oplus 3 = 9,$$

which means that the above points are roots of F, since at each of these points, two monomials give the maximum value.

Since all the coefficients in F are rational and all the monomials are essential, factorization based on roots succeeds as follows:

$$(X \oplus (-1))(X \oplus 1)(X \oplus 3) = X^3 \oplus 3X^2 \oplus 4X \oplus 3.$$

We have the following two results concerning corner roots.

Lemma 2.18. *Let $F, G \in \mathbb{T}[X]$. If $F(a) = G(a)$, for all $a \in \mathbb{T}$, then $\mathcal{Z}_{\mathrm{corn}}(G) = \mathcal{Z}_{\mathrm{corn}}(F)$.*

Proof. We apply contraposition, and let $b \in \mathbb{R}$ such that $b \in \mathcal{Z}_{\mathrm{corn}}(F)$, but $b \notin \mathcal{Z}_{\mathrm{corn}}(G)$. Choose a positive $\varepsilon \in \mathbb{R}$ such that neither F nor G has corner roots at $[b-\varepsilon, b+\varepsilon]$, except for b, which by assumption is a corner root of F. If $F(x) \neq G(x)$, for some $x \in [b, b+\varepsilon]$, we are ready. If $F(x) = G(x)$, for all $x \in [b, b+\varepsilon]$, then F and G go along the same line (with the same slope and the same y-intercept) at $[b, b+\varepsilon]$. This means that F and G has a common monomial (with the same exponent and the same coefficient) determining the value of F and G at $[b, b+\varepsilon]$. Denote this monomial as $F_i = G_i$.

Since b is a corner root of F, it holds that $F(b) = F_i(b) = F_j(b)$, where F_j is another monomial in F. Since F has no corner roots at $[b-\varepsilon, b[$, we can assume that $F(x) = F_j(x) > F_k(x)$, for all $x \in [b-\varepsilon, b[$ and for all other monomials F_k in F ($k \neq j$). Especially, $F_j(b-\varepsilon) > F_i(b-\varepsilon) = G_i(b-\varepsilon)$. Since G has no corner roots at $[b-\varepsilon, b+\varepsilon]$, it holds that $G(x) = G_i(x)$, for all $x \in [b-\varepsilon, b+\varepsilon]$. By collecting these facts together, we obtain

$$F(b-\varepsilon) = F_j(b-\varepsilon) > F_i(b-\varepsilon) = G_i(b-\varepsilon) = G(b-\varepsilon),$$

which proves the (contrapositive) claim. □

Remark. By assuming b and ε, as well as the coefficients of both F and G, to be rational, we can prove the claim: If $F(a) = G(a)$, for all $a \in \mathbb{Q} \cup \{-\infty\}$, then $\mathscr{Z}_{\text{corn}}(G) = \mathscr{Z}_{\text{corn}}(F)$.

Proposition 2.19. *If $F, G \in \mathbb{T}[X]$, then*

$$\mathscr{Z}_{\text{corn}}(F \odot G) = \mathscr{Z}_{\text{corn}}(F) \cup \mathscr{Z}_{\text{corn}}(G).$$

Proof. Write

$$F = \bigoplus_{i=0}^{n} a_i X^i \quad \text{and} \quad G = \bigoplus_{j=0}^{n} b_j X^j,$$

when

$$F \odot G = \bigoplus_{\substack{i,j \in \{0,\ldots,n\} \\ i+j=k}} c_k X^k,$$

where

$$c_k = \max_{\substack{i,j \in \{0,\ldots,n\} \\ i+j=k}} \{a_i b_j\}.$$

Let $x \in \mathscr{Z}_{\text{corn}}(F) \cup \mathscr{Z}_{\text{corn}}(G)$, when, for example, $x \in \mathscr{Z}_{\text{corn}}(F)$. There exists two monomials in F giving the maximal value at x, i.e. $a_i x^i = a_{i'} x^{i'}$, for $i, i' \in \{0, \ldots, n\}$ such that $i \neq i'$. If $b_j X^j$ is a monomial in G giving the maximal value of G at x, then $a_i b_j x^{i+j} = a_{i'} b_j x^{i'+j}$. The expressions on both sides of this equation are maximal summands in $(F \odot G)(x)$, which means that $x \in \mathscr{Z}_{\text{corn}}(F \odot G)$.

Suppose that $x \notin \mathscr{Z}_{\text{corn}}(F) \cup \mathscr{Z}_{\text{corn}}(G)$, which means that $x \notin \mathscr{Z}_{\text{corn}}(F)$ and $x \notin \mathscr{Z}_{\text{corn}}(G)$. Therefore F has a single monomial, say $a_i X^i$, giving a strictly greater value at x than all the other monomials in F, i.e.

$$(a_i X^i)(x) = a_i x^i > a_{i'} x^{i'},$$

for all $i' \in \{1, \ldots, n\}$ such that $i' \neq i$. Similarly, G has a single monomial, say $b_j X^j$, giving a strictly greater value at x than all the other monomials in G, i.e.

$$(b_j X^j)(x) = b_j x^j > a_{j'} x^{j'},$$

for all $j' \in \{1, \ldots, n\}$ such that $j' \neq j$. Therefore

$$(a_i b_j X^{i+j})(x) = a_i b_j x^{i+j} > a_{i'} b_{j'} x^{i'+j'} = (a_{i'} b_{j'} X^{i'+j'})(x),$$

which means that $a_i b_j X^{i+j}$ is such a monomial in $F \odot G$ that gives a strictly greater value at x than all the other monomials in this polynomial. Hence, $x \notin \mathscr{Z}_{\text{corn}}(F \odot G)$. □

Example 2.20. Let $F \in \mathbb{T}[X]$. Based on Proposition 2.19, it holds that $\mathscr{Z}_{\text{corn}}(F) = \mathscr{Z}_{\text{corn}}(F^2)$. As a more concrete case, consider the polynomial $(X \oplus 1)^2 = X^2 \oplus 1X \oplus 2$. Note that Frobenius property holds in $\mathbb{T}$, but not in $\mathbb{T}[X]$. However, the monomial $1X$ is not essential in $X^2 \oplus 1X \oplus 2$, and thus, $(X \oplus 1)(a) = (X^2 \oplus 1X \oplus 2)(a)$, for all $a \in \mathbb{T}$. Therefore, the result $\mathscr{Z}_{\text{corn}}(X \oplus 1) = \mathscr{Z}_{\text{corn}}((X \oplus 1)^2)$ follows also from Lemma 2.18.

The rest of this section is devoted to examples concerning tropical polynomials in two indeterminants. When depicting such a polynomial, we typically draw its corner locus.

Example 2.21. Consider the following real tropical polynomial in two indeterminants:

$$F = X \oplus Y \oplus 1 \in \mathbb{T}[X, Y].$$

As with polynomials in one indeterminant, we again calculate $\max\{X, Y, 1\}$. To find the points, where at least two monomials determinate the maximum value, i.e. the corner roots, we have three choices:

$$1 = x \geq y \quad \text{or} \quad 1 = y \geq x \quad \text{or} \quad x = y \geq 1.$$

These inequations give the corner locus presented in Figure 2.3. The polynomial consists of three essential monomials, and correspondingly the corner locus divides the plane into three regions, where each monomial has greater values than the others.

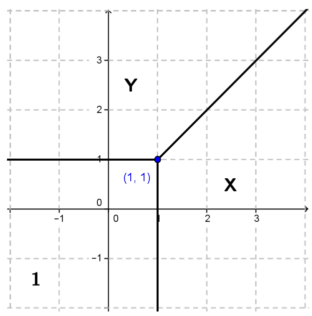


Figure 2.3: The corner locus of the polynomial $F = X \oplus Y \oplus 1$.

Remark. The polynomial $F = X \oplus Y \oplus 1$ is called a *tropical line*, since it is of degree 1.

If a polynomial has more monomials, calcuting the inequations becomes more complicated, as shown in the following example.

Example 2.22. Consider the real tropical polynomial

$$F = -5X^2 \oplus (-5)Y^2 \oplus (-1)XY \oplus X \oplus Y \oplus 0 \in \mathbb{T}[X, Y].$$

It consists of six essential monomials, and its corner locus divides the plane into six regions, as drawn in Figure 2.4.

As an example, we will show how to determinate the region for the monomial $-5X^2$. This is the region, where $-5X^2$ reaches greater values than the other monomials. Therefore it is required to calculate the following inequations:

$$2x-5 \geq 2y-5, \quad 2x-5 \geq x+y-1, \quad 2x-5 \geq x, \quad 2x-5 \geq y, \quad 2x-5 \geq 0.$$

They further imply the inequations:

$$y \leq x, \quad y \leq x-4, \quad x \geq 5, \quad y \leq 2x-5, \quad x \geq 2\frac{1}{2},$$

which give

$$y \leq x-4 \quad \text{and} \quad x \geq 5.$$

In the same way, we can calculate the regions for other monomials, and end up to Figure 2.4.

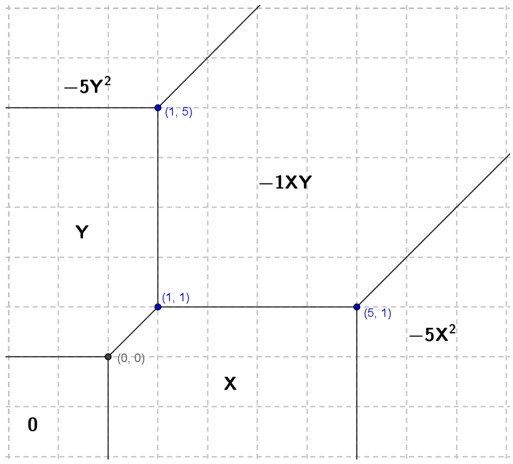


Figure 2.4: The corner locus of the polynomial $F = -5X^2 \oplus (-5)Y^2 \oplus (-1)XY \oplus X \oplus Y \oplus 0$.

We will next consider polynomials with negative exponents and the corner roots of them.

Example 2.23. Consider the polynomial

$$G = -5X \oplus (-5)X^{-1}Y^2 \oplus (-1)Y \oplus 0 \oplus X^{-1}Y \oplus X^{-1} \in \mathbb{T}[X^{\pm 1}, Y^{\pm 1}].$$

When drawing the corner locus of G, we again need to solve certain inequations. In the case of the first monomial, we solve the inequations

$$x-5 \geq -x+2y-5, \quad x-5 \geq y-1, \quad x-5 \geq 0, \quad x-5 \geq -x+y, \quad x-5 \geq -x.$$

They imply the inequations

$$y \leq x, \quad y \leq x-4, \quad x \geq 5, \quad y \leq 2x-5, \quad x \geq 2\frac{1}{2},$$

which are exactly the same inequations as in Example 2.22. Similarly, when calculating the regions for all the other monomials in G, we end up to the same inequations as in the case of F in Example 2.22. Therefore, Figure 2.4 depicts also the corner locus of G. Actually, multiplying G with the monomial X gives F.

Remark. More generally, a polynomial $H \in \mathbb{T}[X_1, \dots, X_n]$ has the same corner roots as any polynomial HH', where $H' \in \mathbb{T}[X_1, \dots, X_n]$ is a monomial. This is easy to understand by considering the calculation of the inequations, as done in Examples 2.22 and 2.23. This observation hints how to eliminate negative exponents. Namely, if $H \in \mathbb{T}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, then by selecting such a monomial $H' \in \mathbb{T}[X_1, \dots, X_n]$ that each indeterminant has an exponent great enough, we have $HH' \in \mathbb{T}[X_1, \dots, X_n]$.

The above kind of multiplication with a monomial preserves the corner roots. However, the values of the multiplied polynomial at these corner roots (as well as at any other point) are different from those of the original polynomial.

2.4 Valuations

Tropical operations are more simple than conventional ones, and thus, evaluation of a polynomial becomes easier. We need only evaluate values of polynomials of degree 1 at most, and take the maximum between them. We will next take a step backwards, and consider how to move from a field to the real tropical semiring. Such a transform exploits a valuation, a map, which typically has a totally ordered group as its target. So we begin with the following definitions concerning orders.

Definition 2.24. Let $(G, \circ)$ be a semigroup, monoid, or a group. It is said that $(G, \preccurlyeq)$ is *partially ordered*, if the relation $\preccurlyeq$ is a partial order in G and if the following condition holds true:

$$a \preccurlyeq b \quad \text{implies} \quad a \circ c \preccurlyeq b \circ c \quad \text{and} \quad c \circ a \preccurlyeq c \circ b,$$

for all $a, b, c \in G$.

Let $(R, +, \cdot)$ be a semiring. It is said that $(R, \preccurlyeq)$ is *partially ordered*, if the relation $\preccurlyeq$ is a partial order in R and if both $(R, +)$ and $(R, \cdot)$ are partially ordered.

Definition 2.25. Let $(G, \circ)$ be a semigroup, monoid, or a group. It is said that $(G, \preccurlyeq)$ is *totally ordered*, if the relation $\preccurlyeq$ is a total order in G and if the following condition holds true:

$$a \preccurlyeq b \quad \text{implies} \quad a \circ c \preccurlyeq b \circ c \quad \text{and} \quad c \circ a \preccurlyeq c \circ b,$$

for all $a, b, c \in G$.

Let $(R, +, \cdot)$ be a semiring. It is said that $(R, \preccurlyeq)$ is *totally ordered*, if the relation $\preccurlyeq$ is a total order in R and if both $(R, +)$ and $(R, \cdot)$ are totally ordered.

Besides the order $\preccurlyeq$, we can define the corresponding strict order $\prec$ in an algebraic structure such that $a \prec b$ holds exactly when $a \preccurlyeq b$ and $a \neq b$, for all elements a and b in the structure in question. However, the strict order behaves well in respect to the operation of a structure only with the cancellative property [3, p. 1].

Proposition 2.26. *Let $(G, \circ)$ be a totally ordered (Abelian) group, monoid, or semigroup, and denote the order as $\preccurlyeq$. The corresponding strict order $\prec$ respects the operation $\circ$, if and only if G is cancellative.*

Proof. It is required to show that

$$a \prec b \quad \text{implies} \quad a \circ c \prec b \circ c \qquad \Longleftrightarrow \qquad a \circ c = b \circ c \quad \text{implies} \quad a = b,$$

for all $a, b, c \in G$.

"$\Rightarrow$" Suppose that $a \circ c = b \circ c$. Therefore $a \circ c \not\prec b \circ c$ and $b \circ c \not\prec a \circ c$. The assumption implies $a \not\prec b$ and $b \not\prec a$, when based on totality $a = b$.

"$\Leftarrow$" Suppose that $a \prec b$. Therefore $a \preccurlyeq b$ and $a \neq b$. Since G is ordered, the former condition implies $a \circ c \preccurlyeq b \circ c$. Based on the assumption, G is cancellative, and thus, the latter condition implies $a \circ c \neq b \circ c$. These results imply together that $a \circ c \prec b \circ c$. □

Remark. The proposition holds true also for semirings (and rings), but then in multiplication, the multiplier (as c above) is required to be non-zero.

The claim holds true for non-Abelian structures, as well. We only required Abelian property to keep the proof simpler. Note that the latter direction of the proof does not require the order to be total.

Example 2.27. Consider the real tropical semiring $(\mathbb{T}, \oplus, \odot)$. Since multiplication is the real addition, it is cancellative and respected by the strict order of real numbers ($<$). On the other hand, addition is not cancellative (as noticed in Example 2.7), and it is not respected by the strict order either.

Now we are ready to move to valuations.

Definition 2.28. Let K be a field and $(G, \circ)$ a totally ordered Abelian group. The map $v : K \to G \cup \{-\infty\}$ is called a *(non-Archimedean) valuation*, if it satisfies the following conditions:

(i) $v(xy) = v(x) \circ v(y)$, for all $x, y \in K$,

(ii) $v(x + y) \leq \max\{v(x), v(y)\}$, for all $x, y \in K$,

(iii) $v(x) = -\infty$, if and only if $x = 0$.

Remark. Here $-\infty$ denotes an element that is smaller than all the elements in G.

Example 2.29. Let K be a field, and consider $\mathbb{R}_{>0}$ as a multiplicative group. If the map

$$|\cdot| : K \to \mathbb{R}_{>0} \cup \{0\}$$

satisfies

(i) $|x| \geq 0$,

(ii) $|x| = 0$, if and only if $x = 0$,

(iii) $|xy| = |x||y|$,

(iv) $|x + y| \leq \max\{|x|, |y|\}$,

for all $x, y \in K$, then it is a valuation.

Remark. The above map is called a *non-Archimedean norm (on K).*

Example 2.30. Let K be a field with a non-Archimedean norm. Then the composition map

$$v : K \to \mathbb{R} \cup \{-\infty\}, \quad v(x) = \begin{cases} \log|x|, & \text{if } x \neq 0 \\ -\infty, & \text{if } x = 0 \end{cases}$$

is a valuation, when considering $\mathbb{R}$ as an additive group. Namely, the points (i) and (iii) required in Definition 2.28 are clear, and based on the properties of the non-Archimedean norm (given in Example 2.29), it holds

$$v(x + y) = \log|x + y| \leq \log\max\{|x|, |y|\} = \max\{\log|x|, \log|y|\},$$

for all non-zero $x, y \in K$, which proves (ii).

The following two definitions and the example after them show how logarithms are related to tropical mathematics.

Definition 2.31. Let K be a field and $F \in K[X_1, \ldots, X_n]$. The *hypersurface* of F is the set

$$\mathcal{Z}_0(F) := \{a \in (K^\times)^n \mid F(a) = 0\}.$$

Definition 2.32. Let K be a field with a non-Archimedean norm. Consider the map

$$\begin{aligned} \operatorname{Log} : (K^\times)^n &\to \mathbb{R}^n \\ (a_1, \ldots, a_n) &\mapsto (\log|a_1|, \ldots, \log|a_n|). \end{aligned}$$

If $F \in K[X_1, \ldots X_n]$, then the set $\operatorname{Log}(\mathcal{Z}_0(F))$ is called the *non-Archimedean amoeba of F.*

Example 2.33. Let K be a field and $v : K \to \mathbb{R} \cup \{-\infty\}$ a (non-Archimedean) valuation. Then the map

$$|\cdot| : K \to \mathbb{R}_{\geq 0}, \quad |a| = e^{v(a)}$$

is a non-Archimedean norm, which can be easily checked by going through the requirements presented in Example 2.29. If $a \in K^{\times}$, then

$$\mathrm{Log}(a) = \log|a| = \log e^{v(a)} = v(a).$$

Therefore, if $F \in K[X_1, \dots X_n]$, then the non-Archimedean amoeba of F is $\mathrm{Log}(\mathcal{Z}_0(F)) = v(\mathcal{Z}_0(F))$.

Valuations have the following properties.

Proposition 2.34. *Let K be a field, $(G, \circ)$ a totally ordered Abelian group with ε as its neutral element, and $v : K \to G \cup \{-\infty\}$ a valuation. Then*

(i) $v(1) = \varepsilon$ *and* $v(-1) = \varepsilon$,

(ii) $v(-a) = v(a)$, *for all* $a \in K$,

(iii) $v(a^{-1}) = (v(a))^{-1}$, *for all* $a \in K$.

Proof. (i) First,

$$v(1) = v(1 \cdot 1) = v(1) \circ v(1),$$

which implies $v(1) = \varepsilon$, since G is a group.

Moreover,

$$v(-1) \circ v(-1) = v((-1) \cdot (-1)) = v(1) = \varepsilon.$$

Since G is totally ordered, it holds either $v(-1) \leq \varepsilon$ or $\varepsilon \leq v(-1)$, where $\leq$ is the order in G. Based on the order condition given in Definition 2.25, the first choice implies

$$v(-1) \circ v(-1) \leq \varepsilon \circ v(-1),$$

which is the same as $\varepsilon \leq v(-1)$. Then antisymmetry implies $v(-1) = \varepsilon$. By assuming the second choice, we again end up to $v(-1) = \varepsilon$ in a similar way.

(ii) If $a \in K$, then by applying (i), we obtain

$$v(-a) = v(-1 \cdot a) = v(-1) \circ v(a) = v(a).$$

(iii) If $a \in K$, then

$$1 = v(1) = v(a \cdot a^{-1}) = v(a) \circ v(a^{-1}),$$

which implies $v(a^{-1}) = (v(a))^{-1}$.

□

Remark. Consider G as an additive group. Then (iii) can be written as $v(a^{-1}) = -v(a)$, for all $a \in K$.

Proposition 2.35. *Let K be a field, G a totally ordered Abelian group, and $v : K \to G \cup \{-\infty\}$ a valuation. Then*

$$\sum_{i=1}^{n} a_i = 0 \quad \textit{implies} \quad v(a_i) = v(a_j) \textit{ for some } i \neq j,$$

where $a_i \in K$ for all $i \in \{1, \dots, n\}$, and $i, j \in \{1, \dots, n\}$.

Proof. Suppose that $\sum_{i=1}^{n} a_i = 0$, and suppose also that the summands a_i are enumerated in the order based on the valuations, as

$$v(a_1) \geq v(a_2) \geq \cdots \geq v(a_n).$$

The equation $\sum_{i=1}^{n} a_i = 0$ can be written both as

$$a_1 = -a_2 - \cdots - a_n \quad \text{and} \quad a_2 = -a_1 - a_3 - \cdots - a_n.$$

By taking valuations, the first equation yields

$$\begin{aligned} v(a_1) &= v(-a_2 - \cdots - a_n) = v(a_2 + \cdots + a_n) \\ &\leq \max\{v(a_2), \dots, v(a_n)\} = v(a_2), \end{aligned}$$

when applying Proposition 2.34 (ii). Similarly, the second equation gives $v(a_2) \leq v(a_1)$. Based on antisymmetry, $v(a_1) = v(a_2)$, and we have found two elements with the same valuation. $\square$

Remark. The above proof shows that these two equal valuations are also the maximum valuations.

2.5 Puiseux series

This section considers a certain field, called the field of Puiseux series, as the domain of a valuation. Before this, we need the following definitions.

Definition 2.36. A *preorder* is a binary relation that is both reflexive and transitive.

Definition 2.37. The set $(S, \leq)$ is said to be a *directed set*, if the relation $\leq$ is a preorder and if every pair of elements of S has an upper bound, meaning that for all $a, b \in S$, there exists $c \in S$ such that $a \leq c$ and $b \leq c$.

Definition 2.38. Let I be a directed set of indices, $\mathcal{C}$ a category, and $\{A_i\}_{i\in I}$ a family of objects in $\mathcal{C}$. Assume that for all $i, j \in I$ such that $i \leq j$, there is a morphism

$$f_{j,i} : A_i \to A_j$$

with the properties

$$f_{i,i} = \mathrm{id} \quad \text{and} \quad f_{k,i} = f_{k,j} \circ f_{j,i},$$

for all $i \leq j \leq k$. Then the set of pairs $(A_i, f_{j,i})$ is a *direct system*. We denote $\mathcal{A} := \{(A_i, f_{j,i})\}$. (For example, if each member A_i in the family $\{A_i\}_{i\in I}$ is a ring and if each $f_{j,i}$ is a ring homomorphism, we speak on a direct of system of rings.)

We will next pay attention to those morphisms that have a common target. Consider the pair $(A, \{f_i\}_{i\in I})$, where $A \in \mathrm{Ob}(\mathcal{C})$ and $\{f_i\}_{i\in I}$ is a family of the morphisms $f_i : A_i \to A$ such that the diagram

$$\begin{tikzcd}
A_i \arrow[rr, "f_{j,i}"] \arrow[dr, "f_i"'] & & A_j \arrow[dl, "f_j"] \\
& A &
\end{tikzcd}$$

commutes for all $i \leq j$, where $i, j \in I$. Then $(A, \{f_i\}_{i\in I})$ is the *direct limit* of the direct system $\mathcal{A}$, if it satisfies the following universal property:

For each pair $(B, \{g_i\}_{i\in I})$, where $B \in \mathrm{Ob}(\mathcal{C})$ and $\{g_i\}_{i\in I}$ is another family of morphisms with a corresponding commutative diagram, there exists a unique morphism $\varphi : A \to B$ such that the diagram

$$\begin{tikzcd}
A_i \arrow[rr, "f_{j,i}"] \arrow[dr, "f_i"'] \arrow[ddr, bend right, "g_i"'] & & A_j \arrow[dl, "f_j"] \arrow[ddl, bend left, "g_j"] \\
& A \arrow[d, "\varphi"] & \\
& B &
\end{tikzcd}$$

commutes.

For short, we say that A is the direct limit of $\mathcal{A}$ and mark $A = \varinjlim A_i$.

In the following definition (and from now on), we denote $\mathbb{N}^*$ for $\mathbb{N}\backslash\{0\}$. The definition assumes the field of Laurent series, i.e. the field of fractions of formal power series over a field.

Definition 2.39. Let $q \in \mathbb{N}^*$, and construct the set of divisors of q:

$$I := \{n \in \mathbb{N}^* \mid n|q\}.$$

Define an order in I based on divisibility as follows:

$$m \leq n \quad \Longleftrightarrow \quad m|n,$$

for all $m, n \in I$. (Clearly, this relation is reflexive and transitive, and thus, a preorder. Moreover, $(I, \leq)$ is a directed set, since the upper bound for each $m, n \in I$ is q.)

Let K be a field, and denote $F_n := K((T^{1/n}))$ for the field of Laurent series with the indeterminant $T^{1/n}$, where $n \in I$. Construct a direct system over $(I, \leq)$, the objects of which are fields F_n $(n \in I)$. Whenever $m \leq n$, for $m, n \in I$, there are morphisms in the direct system, defined as

$$\begin{aligned} f_{n,m} : F_m &\to F_n \\ \sum_{i\in\mathbb{Z}} a_i (T^{\frac{1}{m}})^i &\mapsto \sum_{i\in\mathbb{Z}} a_i (T^{\frac{1}{n}})^i, \end{aligned}$$

where $a_i, b_i \in K$ for all $i \in \mathbb{Z}$. In addition, each $f_{n,n} : F_n \to F_n$ is the identity map.

The direct limit of the direct system is called the *field of Puiseux series* and denoted as $K\{\{T\}\}$.

Remark. Based on the operations of the ring of formal power series, it is easy to see that the morphisms $f_{n,m}$ are indeed homomorphisms for all $m, n \in I$. As a direct limit of fields, the field of Puiseux series is indeed a field.

Puiseux theorem (see e.g. [24, pp. 56–58] (Theorem 2.1.5), or [6, pp. 299–300] (Corollary 13.15)) says that if K is an algebraically closed field of characteric 0, then $K\{\{T\}\}$ is an algebraically closed field.

There is a natural valuation related to Puiseux series, mapping a non-zero Puiseux series to the additive inverse of the lowest exponent of such a term, the coefficient of which is different from zero. If the Puiseux series is zero, i.e. if all the coefficients are zero, the valuation maps to $-\infty$. We will next show that this kind of map is a valuation.

Proposition 2.40. *Let K be a field, when $K\{\{T\}\}$ is a field of Puiseux series over K. Then the map*

$$\begin{aligned} v : K\{\{T\}\} &\to \mathbb{Q} \cup \{-\infty\} \\ \sum_{i=i_0}^{\infty} a_i T^{i/n} &\mapsto \begin{cases} -\frac{i_0}{n}, & \text{if } a_{i_0} \neq 0 \quad (\text{when } \sum_{i=i_0}^{\infty} a_i T^{i/n} \neq 0) \\ -\infty, & \text{if } a_{i_0} = 0 \quad (\text{when } \sum_{i=i_0}^{\infty} a_i T^{i/n} = 0), \end{cases} \end{aligned}$$

where $i_0 \in \mathbb{Z}$, $n \in \mathbb{N}^*$, and $a_i \in K$ for all $i \geq i_0$, is a valuation.

Proof. Assume two Puiseux series, the elements of $K\{\{T\}\}$:

$$F = \sum_{i=i_0}^{\infty} a_i T^{i/n} \quad \text{and} \quad G = \sum_{j=j_0}^{\infty} b_j T^{j/m},$$

where $i_0, j_0 \in \mathbb{Z}$, $n, m \in \mathbb{N}^*$, and $a_i, b_j \in K$ for all $i \geq i_0, j \geq j_0$.

When multiplying two non-zero Puiseux series term by term, all the resulting terms are of the form:

$$a_i T^{\frac{i}{n}} \cdot b_j T^{\frac{j}{m}} = a_i b_j T^{\frac{im+jn}{nm}}.$$

Hence,

$$v(FG) = -\frac{i_0 m + j_0 n}{nm} = -\frac{i_0}{n} + (-\frac{j_0}{m}) = v(F) + v(G).$$

If either of the Puiseux series, say F, is zero, then

$$v(FG) = v(0) = -\infty = -\infty + v(G) = v(F) + v(G).$$

When adding two Puiseux series, it is clear that

$$v(F+G) \leq \max\{-\frac{i_0}{n}, -\frac{j_0}{m}\} = \max\{v(F), v(G)\}.$$

The last valuation requirement is trivial. $\square$

Remark. The above valuation is called an *order valuation.*

The next example considers a polynomial over the field of Puiseux series and how to apply valuation to the coefficients of it.

Example 2.41. Let K be a field, when $K\{\{T\}\}$ is the field of Puiseux series over K. If $F \in K\{\{T\}\}[X]$, then

$$F = \sum_{i=0}^{n} p_i(T) X^i,$$

where $p_i(T) \in K\{\{T\}\}$ for all $i \in \{0, \ldots, n\}$. If v is the order valuation and if we apply it to the coefficients of F, i.e. to the Puiseux series $p_i(T)$, we can denote

$$\tilde{v}(F) := \sum_{i=0}^{n} v(p_i(T)) X^i.$$

In a more concrete case, if

$$F = (8T^5 + 10T^2)X^3 + (7T^{11} + 9T^8) \in \mathbb{C}\{\{T\}\}[X],$$

then

$$\tilde{v}(F) = v(8T^5 + 10T^2)X^3 + v(7T^{11} + 9T^8) = -2X^3 + (-8).$$

2.6 Tropicalization

As mentioned earlier, valuations can be used when moving from usual algebra to tropical one. Such a transition is based on tropicalization, defined as follows.

Definition 2.42. Let K be a field, $(G, \odot)$ a totally ordered Abelian group, $v : K \to G \cup \{-\infty\}$ a valuation, and $F \in K[X_1, \ldots, X_n]$, written as

$$F = \sum_{\boldsymbol{i} \in \mathbb{N}^n} a_{\boldsymbol{i}} X_1^{i_1} \cdots X_n^{i_n},$$

where $a_{\boldsymbol{i}} \in K$ for all $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ such that only finitely many of the coefficients $a_{\boldsymbol{i}}$ differ from zero. Define the map

$$\begin{aligned} \tilde{v} : K[X_1, \ldots, X_n] &\to (G \cup \{-\infty\})[X_1, \ldots, X_n] \\ F &\mapsto \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} v(a_{\boldsymbol{i}}) \odot X_1^{i_1} \odot \cdots \odot X_n^{i_n}, \end{aligned}$$

where $\oplus$ stands for the tropical sum, i.e. maximum between the summands. The tropical polynomial $\tilde{v}(F)$ is called the *tropicalization of F*.

Example 2.43. In the same way as in Example 2.41, suppose that

$$F = (8T^5 + 10T^2)X^3 + (7T^{11} + 9T^8) \in \mathbb{C}\{\{T\}\}[X].$$

If v denotes the order valuation of $\mathbb{C}\{\{T\}\}$ and $\tilde{v}$ is as given in Definition 2.42, then

$$\tilde{v}(F) = v(8T^5 + 10T^2)X^3 \oplus v(7T^{11} + 9T^8) = -2X^3 \oplus (-8).$$

Tropicalization commutes with tropical product in the following sense. In the proof below, we mainly follow [7, p. 6] (Lemma 1.2.1 (i)).

Proposition 2.44. *Let K be a field and $F, G \in K\{\{T\}\}[X]$. Then*

$$\tilde{v}(F \cdot G)(z) = (\tilde{v}(F) \odot \tilde{v}(G))(z),$$

for all $z \in \mathbb{Q} \cup \{-\infty\}$, and

$$\mathcal{Z}_{\text{corn}}(\tilde{v}(F \cdot G)) = \mathcal{Z}_{\text{corn}}(\tilde{v}(F)) \cup \mathcal{Z}_{\text{corn}}(\tilde{v}(G)).$$

Proof. Let $z \in \mathbb{Q} \cup \{-\infty\}$. If $z = -\infty$, then the first claim holds true. Namely, suppose first that both F and G has a (non-zero) constant term, and denote these terms as $c, d \in K\{\{T\}\}$, respectively. Then the constant term of $F \cdot G$ is $c \cdot d$, and thus,

$$\tilde{v}(F \cdot G)(-\infty) = v(c \cdot d) = v(c) \odot v(d) = \tilde{v}(F)(-\infty) \odot \tilde{v}(G)(-\infty).$$

If either F or G has no (non-zero) constant term, then $F \cdot G$ has no (non-zero) constant term, and thus, neither $\tilde{v}(F \cdot G)$ nor $\tilde{v}(F) \odot \tilde{v}(G)$ has a constant term (different from $-\infty$). In this case,

$$\tilde{v}(F \cdot G)(-\infty) = -\infty = -\infty \odot -\infty = \tilde{v}(F)(-\infty) \odot \tilde{v}(G)(-\infty).$$

We can now suppose that z is rational. Therefore we can set $Y := z \odot X$ to obtain

$$(\tilde{v}(H)(Y))(0) = (\tilde{v}(H)(z \odot X))(0) = \tilde{v}(H)(z) = (\tilde{v}(H)(X))(z),$$

for all $z \in \mathbb{Q}$ and for all $H \in K\{\{T\}\}[X]$. Hence

$$\tilde{v}(F \cdot G)(z) = (\tilde{v}(F) \odot \tilde{v}(G))(z)$$

holds, exactly when

$$(\tilde{v}(F \cdot G)(Y))(0) = ((\tilde{v}(F) \odot \tilde{v}(G))(Y))(0)$$

holds, and thus, it requires to prove the latter equation.

Define

$$R := \{a \in K\{\{T\}\} \mid v(a) \leq 0\} \quad \text{and} \quad \mathfrak{m} := \{a \in R \mid v(a) < 0\}.$$

It is easy to see that R is a local ring with $\mathfrak{m}$ as its maximal ideal. Namely, we consider $\mathbb{Q}$ as an additive group, and thus, Proposition 2.34 (iii) implies $v(a^{-1}) = -v(a)$, for all $a \in K\{\{T\}\}$. The only invertible elements $a \in R$ are exactly those, for which $v(a) = 0$, and thus, $\mathfrak{m}$ consists of the non-invertible elements of R.

Assume first that at least one of the exponents of T in the coefficients of F is zero and all the other exponents of T in the coefficients of F are positive. In other words, the least exponent of T occuring in F is exactly zero. Assume that the same holds true for G, too. Therefore the greatest coefficient in $\tilde{v}(F)$ is zero, and the same holds true for $\tilde{v}(G)$. This means that $\tilde{v}(F)(0) = \tilde{v}(G)(0) = 0$, or in other words, $F, G \in R[Y] \backslash \mathfrak{m}[Y]$. We will show that $F \cdot G \in R[Y] \backslash \mathfrak{m}[Y]$.

Note that $\mathfrak{m}[Y] = \mathfrak{m}R[Y]$, and it holds that

$$R[Y]/\mathfrak{m}R[Y] \cong R/\mathfrak{m}[Y],$$

which can be proved by considering the homomorphism

$$R[Y] \to R/\mathfrak{m}[Y], \quad \sum_{i=0}^{n} a_i X^i \mapsto \sum_{i=0}^{n} \pi(a_i) X^i,$$

where $\pi : R \to R/\mathfrak{m}$ is the canonical surjection, and by applying the isomorphism theorem for rings.

Since $\mathfrak{m}$ is maximal, $R/\mathfrak{m}$ is a field, and thus, $R/\mathfrak{m}[Y]$ is an integral domain. Due to the above isomorphism, $R[Y]/\mathfrak{m}R[Y]$ is an integral domain. This implies $\mathfrak{m}R[Y]$ to be a prime ideal, and thus, $R[Y]\backslash\mathfrak{m}[Y]$ is a multiplicative set. Therefore $F \cdot G \in R[Y]\backslash\mathfrak{m}[Y]$, which means that the greatest coefficient in $\tilde{v}(F \cdot G)$ is zero, and thus, $\tilde{v}(F \cdot G)(0) = 0$. As a conclusion,

$$\tilde{v}(F \cdot G)(0) = 0 = 0 \odot 0 = \tilde{v}(F)(0) \odot \tilde{v}(G)(0),$$

which proves the first claim in this case.

Suppose next that the least exponent of T in the coefficients of F differs from zero, and denote it as p. If $p < 0$, define $F' := T^{-p}F$. If $p > 0$, write $F = T^p F'$. In both cases, $F = T^p F'$. Similarly, denote the least exponent of T in the coefficients of G as q, and write $G' := T^{-q}G$, when $G = T^q G'$. Now, the least exponent of T in the coefficients of both F' and G' is exactly zero, and thus, the claim holds for them. Therefore

$$\begin{aligned}
\tilde{v}(F \cdot G)(0) &= \tilde{v}(T^p \cdot F' \cdot T^q \cdot G')(0) = \tilde{v}(T^{p+q} \cdot F' \cdot G')(0) \\
&= -(p+q) \odot \tilde{v}(F' \cdot G')(0) = -p \odot -q \odot \tilde{v}(F')(0) \odot \tilde{v}(G')(0) \\
&= -p \odot \tilde{v}(F')(0) \odot -q \odot \tilde{v}(G')(0) \\
&= \tilde{v}(T^p \cdot F')(0) \odot \tilde{v}(T^q \cdot G')(0) = \tilde{v}(F)(0) \odot \tilde{v}(G)(0),
\end{aligned}$$

which proves the first claim in this case.

Consider finally the second claim. Based on Lemma 2.18 and the remark after it, the first claim implies

$$\mathcal{Z}_{\text{corn}}(\tilde{v}(F) \odot \tilde{v}(G)) = \mathcal{Z}_{\text{corn}}(\tilde{v}(F \cdot G)).$$

Based on Proposition 2.19,

$$\mathcal{Z}_{\text{corn}}(\tilde{v}(F) \odot \tilde{v}(G)) = \mathcal{Z}_{\text{corn}}(\tilde{v}(F)) \cup \mathcal{Z}_{\text{corn}}(\tilde{v}(G)),$$

and thus, $\mathcal{Z}_{\text{corn}}(\tilde{v}(F \cdot G)) = \mathcal{Z}_{\text{corn}}(\tilde{v}(F)) \cup \mathcal{Z}_{\text{corn}}(\tilde{v}(G))$. □

Remark. The above claim holds also for polynomials over any field in several indeterminants [7, p. 6] (Lemma 1.2.1 (i)). However, the above formulation is sufficient for our purposes.

We will next introduce Kapranov's theorem, which gives connection between the roots of a polynomial and the corner roots of the tropicalization of the polynomial. It requires the following proposition describing such a connection in the case of one indeterminant.

Proposition 2.45. *Let K be an algebraically closed field of characteristic* 0 *and $F \in K\{\{T\}\}[X]$. If $b \in \mathcal{Z}_{\text{corn}}(\tilde{v}(F))$, then F has a root $a \in K\{\{T\}\}$ such that $v(a) = b$.*

Proof. Now K is a field, which implies that also $K\{\{T\}\}$ is a field. Therefore we can assume F to be a monic polynomial. We will apply induction on $\deg F$ (which is the same as $\deg \tilde{v}(F)$). If $\deg F = 1$, we can write $F = X - a$ and $\tilde{v}(F) = X \oplus v(a)$, which proves the claim in the base step.

Suppose next that the claim holds true for degrees less than n. Let $\deg F = n$ and $b \in \mathcal{Z}_{\text{corn}}(\tilde{v}(F))$. Since K is an algebraically closed field of characteristic 0, Puiseux theorem [24, pp. 56–58] (Theorem 2.1.5) implies $K\{\{T\}\}$ to be algebraically closed. Therefore we can write $F = (X - c)G$, where $c \in K\{\{T\}\}$ is a root of F and $G \in K\{\{T\}\}[X]$ such that $\deg G = n - 1$. If $v(c) = b$, we are ready. Otherwise Proposition 2.44 implies that $\mathcal{Z}_{\text{corn}}(\tilde{v}(F)) = \mathcal{Z}_{\text{corn}}(\tilde{v}(X - c)) \cup \mathcal{Z}_{\text{corn}}(\tilde{v}(G))$. Since $b \neq v(c)$, it holds that

$$b \notin \mathcal{Z}_{\text{corn}}(\tilde{v}(X - c)) = \mathcal{Z}_{\text{corn}}(X \oplus v(c)) = \{v(c)\}.$$

Since $b \in \mathcal{Z}_{\text{corn}}(\tilde{v}(F))$, it must be $b \in \mathcal{Z}_{\text{corn}}(\tilde{v}(G))$. Now $\deg G = n - 1$, and thus, induction hypothesis implies that $b = v(a)$, for $b \in \mathcal{Z}_{\text{corn}}(\tilde{v}(G)) \subset \mathcal{Z}_{\text{corn}}(\tilde{v}(F))$. $\square$

There are several formulations and proofs for Kapranov's theorem, such as [24, p. 113] and [5, p. 3] (Theorem 2.1.1). The following one gives a constructive proof inspired by [29] and [2]. Note that the proof of [29] is corrected in [2, p. 33] (Theorem 3.15).

Theorem 2.46 (Kapranov)**.** *Let K be an algebraically closed field of characteristic 0, $F \in K\{\{T\}\}[X_1, \ldots, X_n]$, and $v : K\{\{T\}\} \to \mathbb{Q} \cup \{-\infty\}$ the order valuation. Then*

$$v(\mathcal{Z}_0(F)) = \mathcal{Z}_{\text{corn}}(\tilde{v}(F)) \cap \mathbb{Q}^n,$$

where $\tilde{v}(F)$ is the tropicalization of F.

(The set on the left is the non-Archimedean amoeba of F, as mentioned in Example 2.33.)

Proof. Note first that if $\boldsymbol{a} = (a_1, \ldots, a_n) \in K\{\{T\}\}$, we denote

$$v(\boldsymbol{a}) = (v(a_1), \ldots, v(a_n)).$$

Write

$$F = \sum_{\boldsymbol{i} \in \mathbb{N}^n} c_{\boldsymbol{i}} X^{\boldsymbol{i}},$$

where $c_{\boldsymbol{i}} \in K\{\{T\}\}$ for all $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ such that only finitely many of the coefficients $c_{\boldsymbol{i}}$ differ from zero.

"$\subset$" Suppose that $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathcal{Z}_0(F)$, when $v(\boldsymbol{a}) \in v(\mathcal{Z}_0(F))$. (Based on Definition 2.31, we can conclude that F is not a monomial.) The assumption $\boldsymbol{a} \in \mathcal{Z}_0(F)$ means that $F(\boldsymbol{a}) = 0$, and thus, $\sum c_{\boldsymbol{i}} \boldsymbol{a}^{\boldsymbol{i}} = 0$. Based on

Proposition 2.35, this equation implies that there are at least two summands having the same valuations, i.e. $v(c_{\boldsymbol{i}}\boldsymbol{a}^{\boldsymbol{i}}) = v(c_{\boldsymbol{j}}\boldsymbol{a}^{\boldsymbol{j}})$ for some $\boldsymbol{i} \neq \boldsymbol{j}$. Here

$$\begin{aligned} v(c_{\boldsymbol{i}}\boldsymbol{a}^{\boldsymbol{i}}) &= v(c_{\boldsymbol{i}}a_1^{i_1}\cdots a_n^{i_n}) = v(c_{\boldsymbol{i}})v(a_1)^{i_1}\cdots v(a_n)^{i_n} \\ &= v(c_{\boldsymbol{i}})(X_1^{i_1}\cdots X_n^{i_n})(v(a_1),\ldots,v(a_n)) \\ &= \tilde{v}(c_{\boldsymbol{i}}X_1^{i_1}\cdots X_n^{i_n})(v(a_1),\ldots,v(a_n)) \\ &= (\tilde{v}(c_{\boldsymbol{i}}\boldsymbol{X}^{\boldsymbol{i}}))(v(\boldsymbol{a})), \end{aligned}$$

and thus, we can write the equation $v(c_{\boldsymbol{i}}\boldsymbol{a}^{\boldsymbol{i}}) = v(c_{\boldsymbol{j}}\boldsymbol{a}^{\boldsymbol{j}})$ in the form

$$(\tilde{v}(c_{\boldsymbol{i}}\boldsymbol{X}^{\boldsymbol{i}}))(v(\boldsymbol{a})) = (\tilde{v}(c_{\boldsymbol{j}}\boldsymbol{X}^{\boldsymbol{j}}))(v(\boldsymbol{a})).$$

Now $c_{\boldsymbol{i}}\boldsymbol{X}^{\boldsymbol{i}}$ and $c_{\boldsymbol{j}}\boldsymbol{X}^{\boldsymbol{j}}$ are monomials in F, while $\tilde{v}(c_{\boldsymbol{i}}\boldsymbol{X}^{\boldsymbol{i}})$ and $\tilde{v}(c_{\boldsymbol{j}}\boldsymbol{X}^{\boldsymbol{j}})$ are those in $\tilde{v}(F)$. Therefore the above equation means that at least two monomials in $\tilde{v}(F)$ have the same value at $v(\boldsymbol{a})$. According to the proof of Proposition 2.35 (and the remark after it), these monomials give the maximum value, which is the value of whole $\tilde{v}(F)$. Hence, $v(\boldsymbol{a}) \in \mathcal{Z}_{\text{corn}}(\tilde{v}(F))$. Since v is the order valuation, it is clear that $v(\boldsymbol{a}) \in \mathbb{Q}^n$.

"$\supset$" Suppose that $\boldsymbol{b} = (b_1,\ldots,b_n) \in \mathcal{Z}_{\text{corn}}(\tilde{v}(F)) \cap \mathbb{Q}^n$. The aim is to construct $\boldsymbol{a} = (a_1,\ldots,a_n) \in \mathcal{Z}_0(F)$ such that $v(\boldsymbol{a}) = \boldsymbol{b}$.

To apply Proposition 2.45, we will first derive a polynomial in one indeterminant based on F. Consider the exponents of the indeterminants in F i.e. the n-tuples $\boldsymbol{i} = (i_1,\ldots,i_n) \in \mathbb{N}^n$, and denote the greatest exponent occuring in F as m. More precisely,

$$m := \max\{i_k \mid c_{\boldsymbol{i}} \neq 0 \text{ and } k \in \{1,\ldots,n\}\},$$

where $c_{\boldsymbol{i}}$ is a coefficient in F. By selecting $p \in \mathbb{N}^*$ such that $10^p > m$, we can derive from each $\boldsymbol{i} = (i_1,\ldots,i_n)$ a natural number

$$i' := i_1 \cdot (10^p)^{n-1} + i_2 \cdot (10^p)^{n-2} + \cdots + i_{n-1} \cdot (10^p)^1 + i_n \cdot (10^p)^0.$$

Each $\boldsymbol{i}$ produces a unique i', or in other words, there is an injection from $\boldsymbol{i}$ to i'. Namely, if $i' = j'$, then

$$\begin{aligned} &i_1 \cdot (10^p)^{n-1} + i_2 \cdot (10^p)^{n-2} + \cdots + i_{n-1} \cdot 10^p + i_n \\ =&j_1 \cdot (10^p)^{n-1} + j_2 \cdot (10^p)^{n-2} + \cdots + j_{n-1} \cdot 10^p + j_n, \end{aligned}$$

which is the same as

$$\begin{aligned} &10^p(i_1 \cdot (10^p)^{n-2} + i_2 \cdot (10^p)^{n-3} + \cdots + i_{n-1}) + i_n \\ =&10^p(j_1 \cdot (10^p)^{n-2} + j_2 \cdot (10^p)^{n-3} + \cdots + j_{n-1}) + j_n. \end{aligned}$$

Since $10^p > m \geq i_n$ and $10^p > m \geq j_n$, it must be $i_n = j_n$, and thus,

$$i_1 \cdot (10^p)^{n-2} + i_2 \cdot (10^p)^{n-3} + \cdots + i_{n-1} = j_1 \cdot (10^p)^{n-2} + j_2 \cdot (10^p)^{n-3} + \cdots + j_{n-1}.$$

By proceeding in the same way as previously, we can next conclude that $i_{n-1} = j_{n-1}$, and further $i_k = j_k$, for all $k \in \{1, \ldots, n\}$, which means that $\boldsymbol{i} = \boldsymbol{j}$.

Define $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ as

$$\alpha_k := (10^p)^{n-k},$$

for all $k \in \{1, \ldots, n\}$, and $G \in K\{\{T\}\}[X]$ as

$$\begin{aligned} G &:= F(T^{-b_1}X^{\alpha_1}, \ldots, T^{-b_n}X^{\alpha_n}) \\ &= \sum_{\boldsymbol{i} \in \mathbb{N}^n} c_{\boldsymbol{i}} (T^{-b_1}X^{\alpha_1})^{i_1} \cdots (T^{-b_n}X^{\alpha_n})^{i_n}, \\ &= \sum_{\boldsymbol{i} \in \mathbb{N}^n} c_{\boldsymbol{i}} T^{-(b_1 i_1 + \cdots + b_n i_n)} X^{\alpha_1 i_1 + \cdots + \alpha_n i_n}. \end{aligned}$$

Tropicalization of G gives

$$\begin{aligned} \tilde{v}(G) &= \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} v(c_{\boldsymbol{i}} T^{-(b_1 i_1 + \cdots + b_n i_n)}) \odot X^{\alpha_1 i_1 + \cdots + \alpha_n i_n} \\ &= \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} v(c_{\boldsymbol{i}}) \odot (b_1 \cdot i_1 + \cdots + b_n \cdot i_n) \odot X^{\alpha_1 i_1 + \cdots + \alpha_n i_n} \\ &= \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} v(c_{\boldsymbol{i}}) \odot b_1^{i_1} \odot \cdots \odot b_n^{i_n} \odot X^{\alpha_1 i_1} \odot \cdots \odot X^{\alpha_n i_n} \\ &= \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} v(c_{\boldsymbol{i}}) \odot (b_1 \odot X^{\alpha_1})^{i_1} \odot \cdots \odot (b_n \odot X^{\alpha_n})^{i_n}. \end{aligned}$$

The powers above mean tropical exponentiation, although we have not written it visible (as $b_j^{\odot i_j}$). Based on the above expression, we can see that

$$\tilde{v}(G)(0) = \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} v(c_{\boldsymbol{i}}) \odot b_1^{i_1} \odot \cdots \odot b_n^{i_n} = \tilde{v}(F)(\boldsymbol{b}).$$

Since the sum $\alpha_1 i_1 + \cdots + \alpha_n i_n$ (in the powers of X in G) is taken to correspond injectively to $\boldsymbol{i} = (i_1, \ldots, i_n)$, the polynomials F and G have the same number of monomials. Therefore also $\tilde{v}(F)$ and $\tilde{v}(G)$ have the same number of monomials. Moreover, the previous equation reveals that $\boldsymbol{i}$th summand in $\tilde{v}(G)(0)$ is equal to $\boldsymbol{i}$th summand in $\tilde{v}(F)(\boldsymbol{b})$, for all $\boldsymbol{i} \in \mathbb{N}^n$. Since $\boldsymbol{b}$ is a corner root of $\tilde{v}(F)$, there are at least two maximal summands in $\tilde{v}(F)(\boldsymbol{b})$, and since the the summands in $\tilde{v}(F)(\boldsymbol{b})$ and $\tilde{v}(G)(0)$ are the same, there are at least two maximal summands also in $\tilde{v}(G)(0)$. This means that 0 is a corner root of $\tilde{v}(G)$. Therefore Proposition 2.45 implies that G has a root s, which satisfies $v(s) = 0$. By setting $\boldsymbol{a} := (s^{\alpha_1}T^{-b_1}, \ldots, s^{\alpha_n}T^{-b_n})$, we obtain $F(\boldsymbol{a}) = G(s) = 0$. Moreover, $v(\boldsymbol{a}) = \boldsymbol{b}$, since $v(s) = 0$. □

The following two examples clarify Kapranov's theorem in the cases of a single indeterminant and two indeterminants.

Example 2.47. Let K be an algebraically closed field of characteristic 0, and

$$F = T(X - T^2)(X + T^4) = TX^2 + (T^5 - T^3)X - T^7 \in K\{\{T\}\}[X].$$

Clearly, F has the roots T^2 and $-T^4$, i.e. $\mathcal{Z}_0(F) = \{T^2, -T^4\}$.

Assume that v is the order valuation of a Puiseux series, and $\tilde{v}$ is as given in Definition 2.42. Then

$$\tilde{v}(F) = v(T)X^2 \oplus v(T^5 - T^3)X \oplus v(-T^7) = -1X^2 \oplus (-3)X \oplus (-7),$$

which means the same as $\max\{-1 + 2X, -3 + X, -7\}$. Therefore, $\tilde{v}(F)$ has two corner roots: -2 and -4, i.e. $\mathcal{Z}_{\text{corn}}(\tilde{v}(F)) = \{-2, -4\}$.

On the other hand,

$$v(\mathcal{Z}_0(F)) = v(\{T^2, -T^4\}) = \{v(T^2), v(-T^4)\} = \{-2, -4\}.$$

Example 2.48. Let K be an algebraically closed field of characteristic 0, and

$$F = (X - T^2)(Y + T^4) = XY + T^4X - T^2Y - T^6 \in K\{\{T\}\}[X, Y].$$

Now the zero set of F consists of all points (T^2, y) and $(x, -T^4)$, where $x, y \in K\{\{T\}\}$. More precisely,

$$\mathcal{Z}_0(F) = \{(T^2, y) \mid y \in K\{\{T\}\}\} \cup \{(x, -T^4) \mid x \in K\{\{T\}\}\}.$$

Assume that v is the order valuation of a Puiseux series, and $\tilde{v}$ is as given in Definition 2.42. Then

$$\begin{aligned} \tilde{v}(F) &= v(T^0)XY \oplus v(T^4)X \oplus v(-T^2)Y \oplus v(-T^6) \\ &= XY \oplus (-4)X \oplus (-2)Y \oplus (-6). \end{aligned}$$

This gives us certain inequations in a similar way than in Example 2.22. By evaluating them, we can conclude that the corner roots consist of the union of two lines: $x = -2$ and $y = -4$. More precisely,

$$\mathcal{Z}_{\text{corn}}(\tilde{v}(F)) = \{(-2, y) \mid y \in \mathbb{R}\} \cup \{(x, -4) \mid x \in \mathbb{R}\}.$$

On the other hand,

$$\begin{aligned} v(\mathcal{Z}_0(F)) &= v(\{(T^2, y) \mid y \in K\{\{T\}\}\} \cup \{(x, -T^4) \mid x \in K\{\{T\}\}\}) \\ &= \{(v(T^2), v(y)) \mid y \in K\{\{T\}\}\} \cup \{(v(x), v(-T^4)) \mid x \in K\{\{T\}\}\} \\ &= \{(-2, v(y)) \mid v(y) \in \mathbb{Q}\} \cup \{(v(x), -4) \mid v(x) \in \mathbb{Q}\}, \end{aligned}$$

and thus, $v(\mathcal{Z}_0(F)) = \mathcal{Z}_{\text{corn}}(\tilde{v}(F)) \cap \mathbb{Q}^2$.

Consider finally how the proof of Theorem 2.46 goes in this example. Take, for instance, $\boldsymbol{b} = (-2, 1) \in \mathcal{Z}_{\mathrm{corn}}(\tilde{v}(F))$, and choose $\boldsymbol{\alpha} = (10, 1)$. With these selections

$$\begin{aligned} G &= F(T^2X^{10}, T^{-1}X) = T^2X^{10}T^{-1}X + T^4T^2X^{10} - T^2T^{-1}X - T^6 \\ &= TX^{11} + T^6X^{10} - TX - T^6, \end{aligned}$$

and thus,

$$\tilde{v}(G) = -1X^{11} \oplus (-6)X^{10} \oplus (-1)X \oplus (-6).$$

Therefore

$$\tilde{v}(G)(0) = -1 \oplus (-6) \oplus (-1) \oplus (-6) = -1,$$

which means that 0 is a corner root of $\tilde{v}(G)$. The above expression is the same (summand by summand) as $\tilde{v}(F)(-2, 1)$, as told in the proof of Theorem 2.46.

Remark. Based on Kapranov's theorem, we can search for the corner roots of a tropical polynomial instead of searching for the zero set of a polynomial over a Puiseux series.

Chapter 3

Layered tropical semirings

3.1 Bipotency and total order

The previous chapter introduced tropicalization, which enables move the coefficients of a polynomial from the field of Puiseux series to real (rational) numbers. Since calculations in the latter structure are much simpler than in the former one, real tropical mathematics concentrates on the latter one. More generally, the target of a valuation is a totally ordered group, and thus, we will in sequel concentrate on such structures, or even more generally on totally ordered monoids. Since a totally ordered monoid has an equivalent structure to that of a bipotent semiring (which will be seen soon), our main focus from now on will be the algebraic structure of a bipotent semiring.

Elements of a semiring do not necessarily have additive inverses, and thus, the additive neutral element is not so important in semirings than it is in rings. In fact, in the real tropical semiring, the additive neutral element was needed to be explicitly added ($\mathbb{T} = \mathbb{R} \cup \{-\infty\}$). By recalling the definition for a semiring (Definition 2.2) and dropping out everything concerning zero, we obtain the definition below for a semiring without the additive neutral element [17, p. 3], [20, p. 2].

(There is a newer definition for the same concept [21, p. 3], but we prefer the old one, since the new one would cause problems later in the context of layers, as will be explained in the remark after Definition 3.19).

Definition 3.1. The triple $(R, +, \cdot)$ is called a *semiring without zero*, if

(i) $(R, +)$ is an Abelian semigroup,

(ii) $(R, \cdot)$ is a monoid (1 as a typical neutral element),

(iii) multiplication distributes over addition.

A ”semiring without zero” can be written as ”semiring†”.

In the same way as in Definition 2.2, we assume a semiring† to be commutative, if not otherwise mentioned.

Remark. A semiring† becomes a semiring by adding a zero element satisfying

$$0 \cdot a = 0 = a \cdot 0,$$

for all elements a of the semiring†.

Naturally, each semiring is a semiring†. However, if a semiring† contains no zero element, it is called a *proper semiring*†. Consistently with Definition 2.9, a non-zero semiring† is called a *semidomain*†, if it is cancellative (and commutative). If each (non-zero) element in a semiring† has a multiplicative inverse, then the semiring† is called a *semifield*†.

Since each semiring is a semiring†, the latter concept is more general than the former one. We will not take advantage of the missing zero in any proof, instead we will use a semiring† as a generalization of a semiring in this chapter as well as in the next one. Later, in Chapters 5 and 6, we will again need the additive neutral element, and thus, we will finally return to semirings.

Example 3.2. The real tropical semiring, $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, is a semiring†. Because zero element is not required, it would not be necessary to add $-\infty$, so also $(\mathbb{R}, \oplus, \odot)$ is a semiring†.

In sequel, we denote

$$\mathbb{T}^* := (\mathbb{T} \backslash \{-\infty\}, \oplus, \odot) = (\mathbb{R}, \oplus, \odot).$$

Clearly, $\mathbb{T}^*$ is a semidomain†, since its multiplication, which is the addition of $\mathbb{R}$, is cancellative. Moreover, $\mathbb{T}^*$ is a semifield†, since now multiplicative inverses mean additive inverses of $\mathbb{R}$.

Homomorphisms are defined in the obvious way [15, Remark 2.8], as follows.

Definition 3.3. Let R and S be semirings†. The map $f : R \to S$ is called a *(semiring†) homomorphism*, if

(i) $f(a + b) = f(a) + f(b)$,

(ii) $f(ab) = f(a)f(b)$,

(iii) $f(1_R) = 1_S$,

for all $a, b \in R$.

A subsemiring† and an ideal of a semiring† are defined in the way corresponding to those of a ring.

Definition 3.4. Let $(R, +, \cdot)$ be a (commutative) semiring†, and $S \subset R$. It is said that S is a *subsemiring*† of R, if $1 \in S$ and $a+b, ab \in S$ for all $a, b \in S$.

Definition 3.5. Let $(R, +, \cdot)$ be a (commutative) semiring† and $\emptyset \neq I \subset R$. It is said that I is an *ideal* of R, if $a + b \in I$ for all $a, b \in I$, and if $ra \in I$ for all $a \in I$ and $r \in R$.

We will next show that a bipotent semiring† can always be constructed from a totally ordered monoid and vice versa.

Proposition 3.6. *Let $(M, \cdot)$ be a totally ordered monoid. Then $(R, +, \cdot)$ is a bipotent semiring†, where $R = M$ as sets, and $+$ is defined as follows:*

$$a + b = \max\{a, b\},$$

for all $a, b \in R$, where the maximum is determined based on the order of M.

Proof. To prove that $(R, +)$ is an Abelian semigroup, suppose that $a, b \in R$. Now $a + b = \max\{a, b\} \in \{a, b\} \subset R$. It is clear that the addition defined as maximum is associative and that Abelian property is satisfied. It is also clear that $(R, \cdot)$ is a monoid, since $(M, \cdot)$ is such.

To show that the multiplication distributes over addition, denote the order of M as $\leq$, and suppose that $a, b, c \in R$ such that $c \leq b$. Therefore $ac \leq ab$, and

$$a \cdot (b + c) = a \cdot b = a \cdot b + a \cdot c.$$

Bipotency follows from the given definition of addition. □

Remark. The construction of a bipotent semiring requires adding an additive neutral element, denoted as $-\infty$, with the following properties:

(i) $-\infty < a$, for all $a \in R$,

(ii) $-\infty \cdot a = -\infty = a \cdot (-\infty)$, for all $a \in R$.

The opposite fact holds, too: It is possible to construct a totally ordered monoid from a bipotent semiring†.

Proposition 3.7. *Let $(R, +, \cdot)$ be a bipotent semiring†. Then $(R, \cdot)$ is a totally ordered monoid.*

Proof. Trivially, $(R, \cdot)$ is a monoid. Since $(R, +, \cdot)$ is bipotent, it holds that $a + b \in \{a, b\}$, for all $a, b \in R$. Therefore we can define the relation $\leq$ in R based on the addition of $(R, +, \cdot)$ as follows:

$$a \leq b \quad \Longleftrightarrow \quad a + b = b,$$

for all $a, b \in R$. The task is to prove R, equipped with the above order, to be totally ordered.

Antisymmetry can proved as follows. If $a, b \in R$ such that $a \leq b$ and $b \leq a$, then $a + b = b$ and $b + a = a$. Since addition is commutative in R, we obtain $a = b$.

Transitive property can proved as follows. If $a, b, c \in R$ such that $a \leq b$ and $b \leq c$, then $a + b = b$ and $b + c = c$. Therefore

$$a + c = a + (b + c) = (a + b) + c = b + c = c,$$

when using the associative property. Hence, $a \leq c$.

Totality can be proved as follows. If $a, b \in R$ such that $a \nleq b$, then $a + b \neq b$. The bipotency of R means that $a + b \in \{a, b\}$. The only possibility is that $a + b = a$, and thus, $b \leq a$.

We will finally show that the order $\leq$ is preserved under multiplication. Suppose that $a, b, c \in R$ such that $b \leq c$. Therefore $b + c = c$, and

$$ab + ac = a(b + c) = ac,$$

when using the distribute law of $(R, +, \cdot)$. Hence, $ab \leq ac$. $\square$

Based on two previous propositions, we can show that the categories of totally ordered monoids and bipotent semirings† are equivalent [18, p. 11], but before this, we need the following definition [18, p. 9].

Definition 3.8. Let $(M, \leq)$ and $(M', \leq)$ be totally ordered monoids. If $\varphi : M \to M'$ is a monoid homomorphism satisfying the condition:

$$a \leq b \quad \text{implies} \quad \varphi(a) \leq \varphi(b),$$

for all $a, b \in M$, it is called an *order-preserving (monoid) homomorphism.*

Proposition 3.9. *Let $\mathcal{M}$ be a category, the objects of which are totally ordered monoids and the morphisms of which are order-preserving monoid homomorphisms. Let $\mathcal{S}$ be a category, the objects of which are bipotent semirings† and the morphisms of which are semiring† homomorphisms. Then $\mathcal{M}$ and $\mathcal{S}$ are equivalent.*

Proof. Let $(M, \cdot, \leq)$ and $(M', \cdot, \leq)$ be objects of $\mathcal{M}$, and $\varphi : M \to M'$ an order-preserving monoid homomorphism. If $a, b \in M$ such that $a \leq b$, then φ is a semiring† homomorphism, since

$$\varphi(a + b) = \varphi(b) = \varphi(a) + \varphi(b),$$

where addition is defined as in Proposition 3.6. The other properties of a semiring† homomorphim follow directly from those of a monoid homomorphism.

Let $(S, +, \cdot)$ and $(S', +, \cdot)$ be objects of $\mathcal{S}$, and $\psi : S \to S'$ a semiring† homomorphism. Suppose that $a, b \in S$ such that $a \leq b$, where the order $\leq$ is defined as in the proof of Proposition 3.7. Then ψ is an order-preserving monoid homomorphism, since based on semiring† homomorphism properties, it holds

$$\psi(a) + \psi(b) = \psi(a + b) = \psi(b),$$

which implies $\psi(a) \leq \psi(b)$.

Based on these properties between homomorphisms as well as Propositions 3.6 and 3.7, we can prove the equivalence between the categories $\mathcal{M}$ and $\mathcal{S}$. Let $F : \mathcal{M} \to \mathcal{S}$ be a map between these categories. We will first prove F to be a covariant functor. Clearly, we have the map

$$\operatorname{Ob}\mathcal{M} \to \operatorname{Ob}\mathcal{S} \text{ such that } M \mapsto F(M),$$

since as a set M maps to itself and the addition of $F(M)$ can be defined as described in Proposition 3.6. Moreover, we have a map

$$\operatorname{Mor}(M, M') \to \operatorname{Mor}(F(M), F(M')) \text{ such that } f \mapsto F(f),$$

where the morphisms of the domain are order-preserving monoid homomorphisms. Such a map exists, since f actually maps to itself as described earlier with φ. For the same reason, F also preserves compositions of morphisms as well as identity morphisms.

Since the map between morphisms is an identity map, it is bijective, and thus, F is fully faithful. To show F to be essentially surjective suppose that $S \in \operatorname{Ob}\mathcal{S}$. Then $S = F(M)$, for some $M \in \operatorname{Ob}\mathcal{M}$. This follows from the fact that $M = S$ as sets. In other words, we can always find a suitable M by taking $M = S$ as a set and by applying Proposition 3.7.

We have now proved F to be an equivalence between $\mathcal{M}$ and $\mathcal{S}$. □

Example 3.10. Clearly, $(\mathbb{R}, \odot) = (\mathbb{R}, +)$ is a totally ordered monoid. Therefore $(\mathbb{R}, \oplus, \odot)$ is a bipotent semiring†, where $\oplus$ between two real numbers is defined to be the maximum between these numbers.

In Chapter 2, as well as in the previous example, we have considered real tropical semirings, but now we can define a more general structure, as follows.

Definition 3.11. Let $(R, \cdot)$ be a totally ordered monoid. If we define

$$a + b = \max\{a, b\},$$

for all $a, b \in R$, then $(R, +, \cdot)$ is called a *tropical semiring*†.

If there exists in R or if we add to R an additive neutral element, denoted as 0, satisfying

(i) $0 < a$, for all $a \in R$,

(ii) $0 \cdot a = 0 = a \cdot 0$, for all $a \in R$,

then $(R \cup \{0\}, +, \cdot)$ is called a *tropical semiring*.

Remark. Here "bipotent" and "tropical" (i.e. max-plus algebra) mean very much the same, although strictly speaking, the former one is a more general term than the latter one. The similarity follows from the order definition given in the proof of Proposition 3.7, which makes bipotency to mean the same as taking the maximum. Bipotency, as such, could allow e.g. min-plus algebra as well as max-plus algebra.

Example 3.12. Each structure, $(\mathbb{N}, \oplus, \odot)$, $(\mathbb{Z}, \oplus, \odot)$, and $(\mathbb{Q}, \oplus, \odot)$, is a tropical semiring†, where $\odot$ is the normal addition of the structure in question and $\oplus$ means taking the maximum. Two latter structures are tropical semifields†.

3.2 Totally ordered semirings

Orders play an important role in the theory of tropical semirings† and semirings, since addition in these structures is based on a total order. For this reason, we will next pay more attention to orders. Recall Definitions 2.24 and 2.25 for partially and totally ordered semirings and especially the required implication in either of these definitions. The following example shows how it goes when applying it to real numbers.

Example 3.13. Suppose that real numbers $(\mathbb{R}, +, \cdot)$ are equipped with the total order $\leq$ as given in Definition 2.25, instead of the usual order of real numbers. Suppose that $a \in \mathbb{R}$ is positive, when $-a$ is negative. Then $-a \leq a$ implies $a \leq -a$, when multiplied with -1. Since $\leq$ is antisymmetric, it follows that $a = -a$. Hence, the order has the effect that the negative elements vanish, and thus, $\mathbb{R} = \mathbb{R}_+ \cup \{0\}$.

Remark. More generally, the implication requirement, as given in Definition 2.25, makes a ring become a semiring. To see this, suppose that L is a totally ordered ring. Therefore $(L, +)$ is a group, and thus, $-1 \in L$. Now $a \cdot (-1) = -a$ and $-a \cdot (-1) = a$, for all $a \in L$. Since L is totally ordered, we have either $-a \leq a$ or $a \leq -a$. Both cases, after multiplying with -1, lead to $a = -a$, as in the above example. Hence, each element in L merge with its additive inverse, and L becomes a semiring. Especially we have $L = L_+$, when assuming L to be a proper semiring† without the zero element.

Another effect of the order is that zero is the least element in an ordered semiring. Namely, if L is a totally ordered semiring, then we can consider it as a totally ordered multiplicative monoid, as well as, as a bipotent semiring (based on Proposition 3.9). Therefore we have the connection

$$a \leq b \quad \Longleftrightarrow \quad a + b = b,$$

for all $a, b \in L$. It clearly holds $0 + a = a$, for all $a \in L$, and thus, $0 \leq a$, for all $a \in L$. Actually, a (partially) ordered semiring can be defined by requiring that the zero element is the least one, as has been done in [9, p. 1].

Note finally that the terms ”positive” and ”negative” have different meanings in $\mathbb{T}$ and $\mathbb{R}$. If we define an element a of a semiring† to be *positive*, whenever $a > 0$, then all non-zero elements are positive in a tropical semiring† (with max-plus algebra). In min-plus algebra, all non-zero elements are negative, by assuming that we define correspondingly an element a to be *negative*, whenever $a < 0$. For example, in $\mathbb{T}^* = (\mathbb{R}, \oplus, \odot)$, all the elements are positive, even those with the preceeding minus sign. However, $\mathbb{T}$ is not a ring, and the merging effect described above does not concern it.

The next definition gives the standard order relation for a semiring† [15, p. 7].

Definition 3.14. Let L be a semiring†. We define the relation $\geq$ in L, for all $k, l \in L$, as

$$l \geq k \quad \Longleftrightarrow \quad l = k \text{ or } l = k + p \text{ for some } p \in L.$$

Possibly $p = l$. In such case, l is said to be *infinite*. If instead $l + m \neq l$ for all $m \in L$, then $l \in L$ is *finite*. Note that L may include several infinite elements. If an infinite element is unique, it can be denoted as ∞.

We write $l > k$, if $l \geq k$ and $l \neq k$, i.e. if $l = k + p$ for some $p \in L$.

Remark. In the above definition, there is no need to require that the element $p \in L$ is positive (i.e. greater than zero). Namely, if $0 \in L$, then it holds $l = 0 + l$ for all $l \in L$, which means that $l \geq 0$ for all $l \in L$. If $0 \notin L$, then it holds $l = 0 + l$ for all $l \in L \cup \{0\}$, which means that $l \geq 0$ for all $l \in L \cup \{0\}$, i.e. $l > 0$ for all $l \in L$.

As a consequence, if $0 \in L$, then l is infinite, for all $l \in L$.

Proposition 3.15. *Let L be a semiring†, and $\geq$ as given in Definition 3.14. If*

$$l = l + p_1 + p_2 \quad \textit{implies} \quad l = l + p_1,$$

for all $l, p_1, p_2 \in L$, then $(L, \geq)$ is partially ordered.

Proof. We will first prove $\geq$ to be a partial order. The reflexive and transitive properties are easy to see. To show the anti-symmetry property, suppose that $k, l \in L$. If either of the conditions $l \geq k$ or $k \geq l$ implies $k = l$, we are ready. Suppose then that these conditions imply $l = k + p$ and $k = l + q$, respectively, for some $p, q \in L$. Combining these equations yields $l = (l + q) + p$, which based on the assumption implies $l = l + q = k$. (In this case, l and k are infinite.)

We will next show that both $(L, +)$ and $(L, \cdot)$ are ordered. Let $k, l \in L$ such that $l \geq k$. If $l = k$, then $l + m = k + m$ for $m \in L$, and thus, $l + m \geq k + m$. In this case also $lm = km$, and thus, $lm \geq km$. Suppose then that $l = k + p$, for $p \in L$. Now,

$$l + m = k + p + m = (k + m) + p,$$

and thus, $l + m \geq k + m$. Similarly,

$$lm = (k + p)m = km + pm,$$

and thus, $lm \geq km$. $\square$

Remark. If we say L to be a partially ordered semiring†, we assume the order to be $\geq$ as given in Definition 3.14, i.e. $L = (L, \geq)$.

Example 3.16. The tropical semiring† $(\mathbb{T}^*, \oplus, \odot)$ is totally ordered (in the sense of Definitions 2.25 and 3.14).

Proof. To define the order $\geq$ in $\mathbb{T}^*$ according to Definition 3.14, let $k, l \in \mathbb{T}^*$ such that $l \geq k$. This means that either $l = k$ or $l = k \oplus p$, for $p \in \mathbb{T}^*$. In the latter option, bipotency implies $l = k$ or $l = p$, and if the former option does not hold, i.e. $l \neq k$, then $l = p$. Therefore the condition for the order can be written as

$$l \geq k \quad \Longleftrightarrow \quad l = l \oplus k,$$

which is the same order condition as in Proposition 3.7, when constructing a totally ordered monoid from a bipotent semiring†. Based on the proof of this proposition, the above defined relation is a total order in $\mathbb{T}^*$ and $(\mathbb{T}^*, \odot)$ is totally ordered.

It is still required to show that $(\mathbb{T}^*, \oplus)$ is totally ordered. Let $a, b \in \mathbb{T}^*$ such that $a \leq b$. When adding $c \in \mathbb{T}^*$ on both sides, we have three cases. If $c \leq a \leq b$, addition has no effect, and we have $a \leq b$ even after addition. If $a \leq b \leq c$, addition implies $c \leq c$, which is trivially true. If $a \leq c \leq b$, then addition implies $c \leq b$, which holds true in this case. $\square$

We move finally to the ideals of a partially ordered semiring†.

Definition 3.17. Let L be a partially ordered semiring†. We define the subsemiring†

$$L_{\geq 1} := \{l \in L \mid l \geq 1\},$$

and its ideal

$$L_{>1} := \{l \in L_{\geq 1} \mid l = 1 + p \text{ for } p \in L\},$$

with the assumption that the number of elements of L is large enough to ensure that neither of the above sets is empty.

The subsemiring† and ideal properties claimed in the definition will be proved in the following proposition.

Proposition 3.18. *Let L be a partially ordered semiring†. Then $L_{\geq 1} \subset L$ is a subsemiring†, and $L_{>1} \subset L_{\geq 1}$ is an ideal, if not empty sets.*

Proof. Assume that both $L_{\geq 1}$ and $L_{>1}$ are non-empty.

Clearly, $1 \in L_{\geq 1}$. If $a, b \in L_{\geq 1}$, then $a \geq 1$ and $b \geq 1$. This means that $a = 1$ or $a = 1 + p$, and $b = 1$ or $b = 1 + q$, for some $p, q \in L$. In every case $a + b \geq 1$ and $ab \geq 1$, and thus, $a + b, ab \in L_{\geq 1}$.

If $a, b \in L_{>1}$, then $a = 1 + p$ and $b = 1 + q$, for $p, q \in L$, and thus, $a + b = 1 + (1 + p + q)$. Since $1 + p + q \in L$, then $a + b \in L_{>1}$. If, in addition, $l \in L_{\geq 1}$, then $l = 1$ or $l = 1 + m$, for $m \in L$. Now, $al = 1 + p$ or $al = (1 + p)(1 + m) = 1 + (p + m + pm)$, when in both cases $al \in L_{>1}$. □

3.3 The layered structure

A tropical semiring (as given in Definition 3.11) is idempotent. This property has the drawback that in polynomials over a tropical semiring, we cannot recognise whether the maximum value is reached by two or more monomials. Therefore multiplicity of roots cannot be identified. To avoid idempotency, we introduce layers and define addition between two equal elements such that the result (most often) has a layer different from that of the summands.

The following definition for the layer set is taken from [16, p. 4].

Definition 3.19. Let $(L, \geq)$ be a partially ordered semiring†, where $\geq$ is as given in Definition 3.14. If L, in addition, is a directed set, it is called a *sorting semiring*†.

Remark. Most often in this work, we assume that a sorting semiring† is equipped with the usual addition and usual multiplication, instead of the tropical operations. Since $\geq$ is antisymmetric, we can assume that $L = L_+$, in the same way as discussed in Example 3.13 and the remark after it. Most often also, L is discrete, when there is no elements between 0 and 1 (as in the examples below). Therefore $L = L_+ = L_{\geq 1}$, and the multiplicative neutral element 1 is the minimal one.

Since we desire the property $L = L_+ = L_{\geq 1}$, we prefer the formalization of a semiring† as given in Definition 3.1 instead of that given in [21, p. 3]. The difference between these definitions is that the latter and later one (from the year 2014) has a new condition added:

$$a + b = 1, \text{ for some elements } a \text{ and } b \text{ of the semiring†.}$$

Such a condition does not hold for a non-idempotent semiring† with 1 as its minimal element.

Example 3.20. The most usual examples of a sorting semiring† are as follows:

(i) a single layer: $L = \{1\}$,

(ii) two layers: $L = \{1, \infty\}$,

(iii) infinite number of layers: $L = \mathbb{N}^*$.

Naturally, the common upper bound in (i) is 1, while in (ii), it is ∞. In (iii), for each $a, b \in L = \mathbb{N}^*$, the common upper bound is e.g. $\max\{a, b\}$. In all the above cases, L is totally ordered.

Example 3.21. Let $L := \{1, 2, \dots, q\}$ be a sorting semiring†. Suppose that the sum and product of two elements of L are produced as usual, if the result does not exceed q. Otherwise the result is q. Hence, q is infinite. Clearly, q also acts as the common upper bound for each pair of elements.

Example 3.22. The tropical semiring† $\mathbb{T}^*$ is a sorting semiring†. Example 3.16 show that $\mathbb{T}^*$ is totally ordered, and it is easy to see that it is directed. However, it is not a good choice for a sorting semiring†, because it does not remove the additive idempotency property. This will be seen in forth-coming Example 3.37.

We are now ready to give a definition for a layered semiring†, which is modified from [15, Definition 3.6], [16, Definition 4.1], and [18, Definition 5.2]. The modification is textually small, but otherwise significant. It will be discussed in the second remark after the definition.

Definition 3.23. Let $(L, \geq)$ be a sorting semiring†. An *L-layered semiring†* is defined to follow the structure and axioms given below:

The structure is the triple

$$R := (R, L, (\nu_{m,l})),$$

where R is a (commutative) semiring†, consisting of a family $(R_l)_{l \in L}$ of disjoint subsets $R_l \subset R$ such that

$$R = \bigsqcup_{l \in L} R_l,$$

and

$$(\nu_{m,l}) = \{\nu_{m,l} : R_l \to R_m \mid m, l \in L \text{ such that } m \geq l\}$$

is a family of maps, called *(sort) transition maps* satisfying

$$\nu_{l,l} = id_{R_l} \quad \text{and} \quad \nu_{m,l} \circ \nu_{l,k} = \nu_{m,k},$$

for all $k, l, m \in L$ such that $m \geq l \geq k$.

Based on these transition maps, we define a relation, called *ν-equivalence*, as follows: If $a \in R_k$ and $b \in R_l$, for $k, l \in L$, then

$$a \cong_\nu b \quad \Longleftrightarrow \quad \nu_{m,k}(a) = \nu_{m,l}(b),$$

for some $m \in L$ such that $m \geq k, l$. This can be written in a shorter way as $a^\nu = b^\nu$.

The axioms are as follows:

(I) $1_R \in R_1$.

(II) If $a \in R_k$ and $b \in R_l$, then $ab \in R_{kl}$.

(III) The product in R is compatible with sort transition maps:
If $ab \in R_{kl}$, for $a \in R_k$ and $b \in R_l$, then $\nu_{m,k}(a) \cdot \nu_{n,l}(b) = \nu_{mn,kl}(ab)$, for all $m \geq k$ and $n \geq l$.

(IV) Elements of R are (most often) not idempotent:
If $a \in R_k$, then $\nu_{l,k}(a) + \nu_{m,k}(a) = \nu_{l+m,k}(a)$, for all $l, m \geq k$.

(V) The sum in R remains valid at greater layers:
If $a \in R_k, b \in R_l$, and $a + b = c \in R_p$ such that $a \ncong_\nu b$, then $\nu_{m,p}(c) = \nu_{m,k}(a) + \nu_{m,l}(b)$, for all $m \geq k, l, p$.

(VI) R is *supertropical*:
If $a \in R_k$ and $b \in R_l$ such that $a \cong_\nu b$, then $a + b \in R_{k+l}$ with $a + b \cong_\nu a$. If moreover k is infinite, then $a + b = a$.

(VII) R is *ν-bipotent*:
If $a, b \in R$ such that $a \ncong_\nu b$, then $a + b \in \{a, b\}$.

If a non-zero L-layered semiring† is multiplicatively cancellative, it is called an *L-layered semidomain*†.

If the sorting semiring† is understood or is meaningless, we can speak on a *layered semiring*† or a *layered semidomain*†.

Remark. The subset R_1 is called the *tangible layer*, and its elements are called *tangible elements*. The other subsets are called *ghost layers*, and their elements are called *ghost elements*. The tangible layer R_1 is a special one, it is assumed to have the structure of a totally ordered multiplicative monoid. Thus, based on Proposition 3.9, it can be viewed as a bipotent semiring†.

Sort transition maps can be assumed to be surjective, and thus, all elements of R are ν-equivalent with a tangible element.

Due to axiom (IV), the elements of R are additively idempotent only at an infinite layer. However, axiom (VII) (ν-bipotency) permits max-plus-algebra between unequal elements.

Recall that the sorting semiring† L is a directed set. If $\mathcal{C}$ is the category of sets, then the layers form a family of objects in $\mathcal{C}$. Therefore, based on the properties of the sort transition maps, the set of pairs $(R_m, (\nu_{m,l}))$, where $l, m \in L$, form a direct system. By defining

$$R_\infty \ := \ R/\cong_\nu \ = \ (\bigsqcup_{k \in L} R_k)/\cong_\nu,$$

we can prove R_∞ to be the direct limit of the direct system. Actually, in [16], R_∞ is defined to be the direct limit of the direct system described above.

Remark. We have slightly modified axiom (V) by restricting it to the cases, where $a \ncong_\nu b$. Originally, in [16, p. 7], it was given as follows.

(V') If $a \in R_k, b \in R_l$, and $a + b = c \in R_p$, then $\nu_{m,p}(c) = \nu_{m,k}(a) + \nu_{m,l}(b)$, for all $m \geq k + l$.

The reason for the modification is that the above formulation leads to an undesired consequence, when combining it with axiom (IV). Namely, if $a \in R_1$, then axiom (IV) gives

$$a + a = \nu_{1,1}(a) + \nu_{1,1}(a) = \nu_{1+1,1}(a).$$

By applying axiom (V') to the above equation, we obtain

$$\nu_{m,1}(a) = \nu_{m,1+1}(\nu_{1+1,1}(a)) \overset{\text{(V')}}{=} \nu_{m,1}(a) + \nu_{m,1}(a) = \nu_{m+m,1}(a),$$

for all $m \geq 1+1$. The above equation implies $m = m + m$ for all $m \geq 1+1$, which means that all $m > 1$ are infinite. If especially an infinite element is unique, as defined in [16, p. 2] and assumed in [18, p. 10], the layered semiring† degenerates into the case, where the number of layers is two, at most.

For the above reason, we prefer axiom (V) as given in Definition 3.23. Although it lacks the case, where the summands are ν-equivalent, we can conclude this case from the other axioms, as follows.

Lemma 3.24. *Let L be a sorting semiring† and R an L-layered semiring†. If $k, l \in L$ and $a \in R_k, b \in R_l$ such that $a \cong_\nu b$, then*

$$\nu_{m,k}(a) + \nu_{n,l}(b) = \nu_{m+n,k+l}(a + b),$$

for all $m, n \in L$ such that $m, n \geq k, l$.

Proof. By definition, the assumption $a \cong_\nu b$ implies $\nu_{p,k}(a) = \nu_{p,l}(b)$, for some $p \in L$ such that $p \geq k, l$. On the other hand, axiom (VI) implies $a + b \in R_{k+l}$ and $a + b \cong_\nu a$ (as well as $a + b \cong_\nu b$). These conditions imply further $\nu_{p',k+l}(a + b) = \nu_{p',k}(a)$, for some $p' \in L$ such that $p' \geq k + l$.

If $m, n \in L$ such that $m, n \geq p$, then

$$\begin{aligned} \nu_{m,k}(a) + \nu_{n,l}(b) &= \nu_{m,p}(\nu_{p,k}(a)) + \nu_{n,p}(\nu_{p,l}(b)) \\ &= \nu_{m,p}(\nu_{p,k}(a)) + \nu_{n,p}(\nu_{p,k}(a)) \\ &\overset{\text{(IV)}}{=} \nu_{m+n,p}(\nu_{p,k}(a)) \\ &= \nu_{m+n,k}(a) \\ &= \nu_{m+n,p'}(\nu_{p',k}(a)) \\ &= \nu_{m+n,p'}(\nu_{p',k+l}(a + b)) \\ &= \nu_{m+n,k+l}(a + b). \end{aligned}$$

□

Besides ν-equivalence introduced in Definition 3.23, we define another relation for a layered semiring†. We follow [15, p. 8] (Definition 2.18) to base it on ν-equivalence, as follows.

Definition 3.25. Let R be a layered semiring†. We define the relation

$$a \leq_\nu b \quad \Longleftrightarrow \quad a + b \cong_\nu b,$$

for all $a, b \in R$. This can be written in a shorter way as $a^\nu + b^\nu = b^\nu$.

Furthermore, we write $a <_\nu b$, if $a \leq_\nu b$ and $a \ncong_\nu b$.

Remark. There are other formulations for the same relation. In [16, p. 7] and [18, p. 15], it is defined as follows:

Let L be a sorting semiring† and R an L-layered semiring†. If $a \in R_k$ and $b \in R_l$, for $k, l \in L$, then

$$a \leq_\nu b \quad \Longleftrightarrow \quad \nu_{m,k}(a) + \nu_{m,l}(b) = \nu_{m,l}(b),$$

for some $m \in L$ such that $m \geq k, l$.

If $a \ncong_\nu b$, the above definition is equivalent to Definition 3.25. Namely, the ν-equivalence $a + b \cong_\nu b$ is the same as $\nu_{n,k+l}(a + b) = \nu_{n,l}(b)$, for some $n \in L$ such that $n \geq k + l$. Based on axiom (V), this is the same as $\nu_{n,k}(a) + \nu_{n,l}(b) = \nu_{n,l}(b)$. Clearly, $\nu_{n,l}(b) \cong_\nu \nu_{m,l}(b)$, and thus, by taking e.g. the maximum between m and n, we end up to the above definition (when not paying attention to the difference between the assumptions $m \geq k, l$ and $n \geq k + l$).

In the case, where $a \cong_\nu b$, the above definition leads to an undesired consequence. Namely, if $a \cong_\nu b$, then $\nu_{p,k}(a) = \nu_{p,l}(b)$, for some $p \in L$ such that $p \geq k, l$, when the above definition gives further

$$\begin{aligned} \nu_{m,l}(b) = \nu_{m,k}(a) + \nu_{m,l}(b) &= \nu_{m,p}(\nu_{p,k}(a)) + \nu_{m,p}(\nu_{p,l}(b)) \\ &= \nu_{m+m,p}(\nu_{p,l}(b)) = \nu_{m+m,l}(b), \end{aligned} \tag{3.1}$$

when assuming that $m \geq p$. This equation implies $m = m + m$, which means that m is infinite. We will later see (in Proposition 3.27) that $(R, \leq_\nu)$ is totally ordered. Therefore, the condition $a \leq_\nu b$ is not always false, and thus, we can conclude L to always include at least one infinite element.

We wish not restrict ourselves to such a situation, where the sorting semiring† must include an infinite element, and thus, we prefer Definition 3.25, which does not lead to such an undesired conclusion. Namely, if $a \cong_\nu b$, then $\nu_{p,k}(a) = \nu_{p,l}(b)$, for some $p \in L$ such that $p \geq k, l$. By using now Definition 3.25, $a \leq_\nu b$ means that $a + b \cong_\nu b$, which is the same as $\nu_{m,k+l}(a + b) = \nu_{m,l}(b)$, for some $m \in L$ such that $m \geq k + l$. Based on Lemma 3.24,

$$\nu_{m',k}(a) + \nu_{m'',l}(b) = \nu_{m,k+l}(a + b) = \nu_{m,l}(b),$$

for $m', m'' \in L$ such that $m', m'' \geq k, l$ and $m' + m'' = m$. The above equation is an equivalent condition for $a \leq_\nu b$ to that given in Definition 3.25. By using it in a similar equation chain as done in (3.1), we have

$$\begin{aligned}\nu_{m,l}(b) &= \nu_{m',k}(a) + \nu_{m'',l}(b) = \nu_{m',p}(\nu_{p,k}(a)) + \nu_{m'',p}(\nu_{p,l}(b)) \\ &= \nu_{m'+m'',p}(\nu_{p,l}(b)) = \nu_{m,l}(b),\end{aligned}$$

when assuming that $m' \geq p$ and $m'' \geq p$.

Note finally that $a \cong_\nu b$ implies $a \leq_\nu b$. Namely, if $a \cong_\nu b$, then axiom (VI) implies $a + b \cong_\nu b$, which means that $a \leq_\nu b$.

3.4 Properties of layered semirings

We start by paying attention to the relations $\cong_\nu$ given in Definition 3.23 and $\leq_\nu$ given in Definition 3.25. We consider first the former one.

Proposition 3.26. *Let L be a sorting semiring† and R an L-layered semiring†. Relation $\cong_\nu$ given in Definition 3.23 is an equivalence relation respecting addition and multiplication in the following sense: If $a \cong_\nu b$ and $c \cong_\nu d$, then $a + c \cong_\nu b + d$ and $ac \cong_\nu bd$, for all $a, b, c, d \in R$.*

Proof. It is clear that $\cong_\nu$ is reflexive and symmetric. To show transitive property to hold, suppose that $a \cong_\nu b$ and $b \cong_\nu c$, for $a \in R_k, b \in R_l, c \in R_p$, where $k, l, p \in L$. The assumed ν-equivalences mean, respectively,

$$\nu_{m,k}(a) = \nu_{m,l}(b) \quad \text{and} \quad \nu_{n,l}(b) = \nu_{n,p}(c),$$

for $m, n \in L$ such that $m \geq k, l$ and $n \geq l, p$. Since $\nu_{m,l}(b) \cong_\nu \nu_{n,l}(b)$, we can take the maximum between m and n, say m, to obtain $\nu_{m,l}(b) = \nu_{m,p}(c)$. This implies $\nu_{m,k}(a) = \nu_{m,p}(c)$, which means that $a \cong_\nu c$.

To prove the remaining two assertions, assume that $a \cong_\nu b$ and $c \cong_\nu d$, for $a \in R_k, b \in R_{k'}, c \in R_l, d \in R_{l'}$, where $k, k', l, l' \in L$. Therefore

$$\nu_{m,k}(a) = \nu_{m,k'}(b) \quad \text{and} \quad \nu_{n,l}(c) = \nu_{n,l'}(d),$$

for $m, n \in L$ such that $m \geq k, k'$ and $n \geq l, l'$.

Multiplying side by side two equations above gives

$$\nu_{mn,kl}(ac) = \nu_{m,k}(a)\nu_{n,l}(c) = \nu_{m,k'}(b)\nu_{n,l'}(d) = \nu_{mn,k'l'}(bd),$$

when applying axiom (III). This proves that $ac \cong_\nu bd$.

Adding side by side the same equations gives

$$\nu_{m,k}(a) + \nu_{n,l}(c) = \nu_{m,k'}(b) + \nu_{n,l'}(d). \tag{3.2}$$

If $a \cong_\nu c$, then based on transitivity of ν-equivalence, $b \cong_\nu d$. Similarly, if $b \cong_\nu d$, then $a \cong_\nu c$. Therefore $a \cong_\nu c$, exactly when $b \cong_\nu d$.

Suppose first that $a \cong_\nu c$ and $b \cong_\nu d$. By applying Lemma 3.24 to the both sides of the equation in (3.2), we obtain

$$\nu_{m+n,k+l}(a+c) = \nu_{m+n,k'+l'}(b+d),$$

which means that $a + c \cong_\nu b + d$.

Suppose next that $a \ncong_\nu c$ and $b \ncong_\nu d$. Decompose the equation in (3.2) into two equations, as

$$\nu_{m,k}(a) + \nu_{n,l}(c) = e \quad \text{and} \quad e = \nu_{m,k'}(b) + \nu_{n,l'}(d),$$

where $e \in R_t$, for $t \in L$. By applying axiom (V) to both of the above equations, we obtain

$$\nu_{p,m}(\nu_{m,k}(a)) + \nu_{p,n}(\nu_{n,l}(c)) = \nu_{p,t}(e),$$

and

$$\nu_{q,t}(e) = \nu_{q,m}(\nu_{m,k'}(b)) + \nu_{q,n}(\nu_{n,l'}(d)),$$

for all $p, q \in L$ such that $p \geq m, n, t$ and $q \geq m, n, t$. Hence, it is possible to select $p = q$. These equations can now be put together as

$$\nu_{p,k}(a) + \nu_{p,l}(c) = \nu_{p,k'}(b) + \nu_{p,l'}(d).$$

By applying axiom (V) to the both sides of the above equation, we obtain

$$\nu_{p,p'}(a+c) = \nu_{p,q'}(b+d),$$

where $p' \in L$ is the layer of $a+c$ and $q' \in L$ is the layer of $b+d$. (Actually $p' \in \{k, l\}$ and $q' \in \{k', l'\}$.) The above equation means that $a + c \cong_\nu b + d$. $\square$

Proposition 3.27. *Let L be a sorting semiring† and R an L-layered semiring†. Then $(R, \leq_\nu)$ is totally ordered, where $\leq_\nu$ is as given in Definition 3.25. (The order is taken on equivalence classes).*

Proof. The task is to prove the following properties for $\leq_\nu$:

(1) It is a total order (satisfying antisymmetry, transitivity, and totality).

(2) It respects multiplication: If $a, b, c \in R$ such that $a \leq_\nu b$, then $ac \leq_\nu bc$.

(3) It respects addition: If $a, b, c \in R$ such that $a \leq_\nu b$, then $a + c \leq_\nu b + c$.

This is done as follows.

(1) To show transitive property, let $a, b, c \in R$ such that $a \leq_\nu b$ and $b \leq_\nu c$. Based on Definition 3.25, we obtain $a + b \cong_\nu b$ and $b + c \cong_\nu c$. Based

on Proposition 3.26, ν-equivalence is a transitive relation respecting addition, and thus,

$$a + c \cong_\nu a + (b + c) = (a + b) + c \cong_\nu b + c \cong_\nu c.$$

Hence, $a \leq_\nu c$.

To show antisymmetry, let $a, b \in R$ such that $a \leq_\nu b$ and $b \leq_\nu a$. This means that $a + b \cong_\nu b$ and $b + a \cong_\nu a$. Based on Abelian property of addition and transitive property proved just previously, we have $a \cong_\nu b$.

To show totality, let $a, b \in R$ such that $a \not\leq_\nu b$. Based on Definition 3.25, this means that $a + b \ncong_\nu a$, which implies further $a + b \neq a$. It cannot be $a \cong_\nu b$, since otherwise axiom (VI) would imply $a + b \cong_\nu a$, a contradiction. Therefore, it must be $a \ncong_\nu b$, when axiom (VII) implies $a+b \in \{a, b\}$. As concluded just previously, $a+b \neq a$, and thus, the only possibility is $a + b = b$. Therefore $a + b \cong_\nu b$, which means that $a \leq_\nu b$.

(2) The next task is to show $\leq_\nu$ to respect the multiplication of R. Let $a, b, c \in R$ such that $a \leq_\nu b$. This means that $a + b \cong_\nu b$. Based on Proposition 3.26, ν-equivalence respects multiplication, and thus, $ac + bc \cong_\nu bc$. Hence $ac \leq_\nu bc$.

(3) We will finally show $\leq_\nu$ to respect the addition of R. Let $a, b, c \in R$ such that $a \leq_\nu b$. Therefore $a + b \cong_\nu b$. Based on totality, proved at point (1), we have three choices for the position of c with respect to a and b:

$$c \leq_\nu a \leq_\nu b \quad \text{or} \quad a \leq_\nu c \leq_\nu b \quad \text{or} \quad a \leq_\nu b \leq_\nu c.$$

Consider the first choice first. It implies both

$$c + a \cong_\nu a \quad \text{and} \quad c + b \cong_\nu b,$$

where latter one follows from the transitive property of $\leq_\nu$. Based on Proposition 3.26, ν-equivalence respects addition, and thus, we can add the above ν-equivalences side by side. This gives

$$a + c + c + b \cong_\nu a + b \cong_\nu b \cong_\nu c + b,$$

by recalling the assumption $a+b \cong_\nu b$ and by applying transitive property of ν-equivalence with it. Therefore $a + c \leq_\nu b + c$.

Consider next the second choice. We have now both

$$a + c \cong_\nu c \quad \text{and} \quad c + b \cong_\nu b,$$

By adding again both ν-equivalences side by side, we obtain directly

$$a + c + c + b \cong_\nu c + b.$$

Hence, $a + c \leq_\nu b + c$.

Consider finally the third choice. In this case, we have

$$a + c \cong_\nu c \quad \text{and} \quad b + c \cong_\nu c,$$

when transitive property of ν-equivalence gives $a + c \cong_\nu b + c$. Hence $a + c \leq_\nu b + c$.

□

We will next pay attention to the axioms related to the operations in a layered semiring†. The axioms concerning multiplication given in Definition 3.23 are clear, and they do not depend on the values of the multipliers. The axioms concerning addition are more complicated, and thus, we will clarify them in the following example.

Example 3.28. Let L be a sorting semiring† and R an L-layered semiring†, with $k, l \in L$, and $a \in R_k$, $b \in R_l$. We will consider the sum $a + b$ in different cases.

Assume first that $a <_\nu b$. Therefore $a \leq_\nu b$ and $a \not\cong_\nu b$. Based on axiom (VII), the latter condition implies $a + b = a$ or $a + b = b$. If it was $a + b = a$, then $a + b \cong_\nu a$, which means that $b \leq_\nu a$. But this is a contradiction to $a <_\nu b$. Hence, the only possibility is that $a + b = b$. Similarly, if $b <_\nu a$, then $a + b = a$.

If $a = b$, then $k = l$, and

$$a + b = a + a = \nu_{k,k}(a) + \nu_{k,k}(a) = \nu_{k+k,k}(a),$$

which is the same as $\nu_{l+l,l}(b)$. This follows from axiom (IV).

Suppose next that $a \cong_\nu b$ but $a \neq b$. This means that $\nu_{m,k}(a) = \nu_{m,l}(b)$, for some $m \in L$ such that $m \geq k, l$. Suppose that m is the smallest layer, where the equation holds. If $m = \max\{k, l\}$, say $m = l$, then

$$\nu_{m,k}(a) = \nu_{m,l}(b) = \nu_{l,l}(b) = b,$$

and thus,

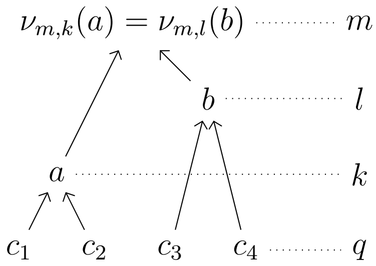
$$a + b = \nu_{k,k}(a) + \nu_{m,k}(a) = \nu_{k+m,k}(a) = \nu_{k+l,k}(a).$$

Similarly, if $m = k$, then $a + b = \nu_{k+l,l}(b)$.

Otherwise (if $m > k, l$), Definition 3.23 gives no calculation rule for $a + b$. Namely, we cannot find a common element $c \in R_q$, where $q \in L$ such that $k, l \geq q$, to express a and b as

$$a = \nu_{k,q}(c) \quad \text{and} \quad b = \nu_{l,q}(c).$$

The situation can be depicted as follows:



The elements c_1, c_2, c_3, and c_4 at layer q illustrate different choices for c, and the upward arrows are sort transition maps. Therefore

$$a = \nu_{q,k}(c_1) = \nu_{q,k}(c_2) \quad \text{and} \quad b = \nu_{q,l}(c_3) = \nu_{q,l}(c_4),$$

but since all elements c_1, c_2, c_3, and c_4 are distinct, there is no element that would be the common preimage for the above sort transition maps. However, such an element is necessary in applying axiom (IV). In this case, we only know that $a + b \cong_\nu a \cong_\nu b$.

Actually, the above situation gives different choices for the sum $a + b$. One possible choice is as follows. If $n \in L$ such that $n \geq k + l$, then based on the composition rules of transition maps

$$\nu_{n,k}(a) = \nu_{n,k+l}(\nu_{k+l,k}(a)),$$

which implies $a \cong_\nu \nu_{k+l,k}(a)$. Based on Lemma 3.26, ν-equivalence is transitive, and thus, $a + b \cong_\nu \nu_{k+l,k}(a)$. Since based on axiom (VI), $a + b \in R_{k+l}$, it is possible to choose $a + b = \nu_{k+l,k}(a)$. Another possible choice would be $a + b = \nu_{k+l,l}(b)$. However, if $m \in L$ is the least layer, where a and b become the same and if $m > k + l$, then $\nu_{k+l,k}(a) \neq \nu_{k+l,l}(b)$, although $\nu_{k+l,k}(a) \cong_\nu \nu_{k+l,l}(b)$.

By collecting the above results together and by making the decision described above, we have

$$a + b = \begin{cases} a, & \text{if } a >_\nu b, \\ b, & \text{if } b >_\nu a, \\ \nu_{k+l,k}(a), & \text{if } a \cong_\nu b. \end{cases}$$

By using the shorter notation, this is the same as

$$a + b = \begin{cases} a, & \text{if } a >_\nu b, \\ b, & \text{if } b >_\nu a, \\ a^\nu, & \text{if } a \cong_\nu b. \end{cases}$$

We will finally consider ideals of a layered semiring†.

Proposition 3.29. *Let L be a sorting semiring† and R an L-layered semiring†. If $l \in L$ such that l is not the greatest element of L, then the subsets*

$$R_{\geq l} := \bigcup_{m \geq l} R_m \quad \textit{and} \quad R_{>l} := \bigcup_{m > l} R_m$$

are ideals of R.

Proof. The assumption ensures that $R_{>l}$ is not the empty set.

We show that $R_{>l}$ is an ideal of R. Let $a, b \in R_{>l}$. If $a \not\cong_\nu b$, then based on axiom (VII), $a + b \in \{a, b\} \subset R_{>l}$. Suppose next that $a \cong_\nu b$. Since $a, b \in R_{>l}$, it holds that $a \in R_m$ and $b \in R_{m'}$, where $m > l$ and $m' > l$. Based on supertropicality axiom, $a+b \in R_{m+m'} \subset R_{>l}$. Suppose finally that $r \in R$. By assuming that 1 is the smallest element in a sorting semiring†, we have $r \in R_{\geq 1}$. Based on axiom (II), $ar \in R_{\geq m} \subset R_{>l}$.

The other subset can be proved to be an ideal in a similar way. □

Remark. Ghost layers serves as ideals of a layered semiring†. Especially, the ideal $R_{>1}$ is called a *ghost ideal.*

3.5 Examples of layered semirings

3.5.1 A single layer

The next example shows that the real tropical semiring can be seen as a layered semiring†.

Example 3.30. Consider $L := \{1\}$ as a sorting semiring†. Since there is only one layer, there are no transition maps, except for identity maps. Since L has only a single element, we assume that

$$1 + 1 = 1 = 1 \cdot 1.$$

In fact, 1 is infinite.

As an example of this kind of situation, let $R := (\mathbb{T}^*, \{1\}, \{\nu_{1,1}\})$. Since the only transition maps are identity maps, ν-equivalence actually means the usual equality. Now, ν-bipotency requirement implies addition to be taking the maximum. This holds true also when adding equal elements:

$$a + a = \nu_{1,1}(a) + \nu_{1,1}(a) = \nu_{1+1,1}(a) = \nu_{1,1}(a) = a,$$

for all $a \in \mathbb{T}^*$. Multiplication means the usual multiplication of $\mathbb{T}^*$, i.e. the usual addition of $\mathbb{R}$. Therefore it is easy to see that

$$R = (\mathbb{T}^*, \{1\}, \{\nu_{1,1}\}) \cong (\mathbb{T}^*, \oplus, \odot).$$

Since all transition maps are identity maps, the axioms in Definition 3.23 become trivial. Hence, R can be seen as an L-layered semiring†.

3.5.2 Two layers

Example 3.31. Let $L := \{1, \infty\}$ be a sorting semiring† with the following tables for addition and multiplication:

$+$	1	∞
1	∞	∞
∞	∞	∞

$\cdot$	1	∞
1	1	∞
∞	∞	∞

Furthermore, we construct an L-layered semiring† R with the following disjoint sets: $R_1 := F^\times$, the invertible elements of a field F, and $R_\infty := \mathcal{G}$, a totally ordered (multiplicative) group. In other words, $F^\times$ comprises the tangible elements, and $\mathcal{G}$ the ghost ones. We consider $F^\times$ as a multiplicative group, when we forget the original addition of F. We will define below a new addition fulfilling the layered tropical requirements.

Since we have only two layers, there exists only one transition map (besides the identity maps), and thus, we need no indices. This transition map is given as

$$\nu : F^\times \to \mathcal{G},$$

and it is supposed to be surjective, when all the elements of $\mathcal{G}$ are of the form $\nu(a)$, marked as a^ν, where $a \in F^\times$. We can extend ν to have whole $F^\times \sqcup \mathcal{G}$ as its domain, when for the elements of $\mathcal{G}$, ν is an identity map. Thus, $\nu^n = \nu$ for all $n \in \mathbb{N}^*$.

To make the disjoint union $F^\times \sqcup \mathcal{G}$ an L-layered semiring†, we need to define a mixed multiplication in $F^\times \sqcup \mathcal{G}$, based on the original ones of both structures. This is done in the following way:

$$a \cdot b^\nu = a^\nu \cdot b = (ab)^\nu, \tag{3.3}$$

where $a, b \in F^\times$ and $a^\nu, b^\nu \in \mathcal{G}$. This implies

$$(ab)^\nu = ((ab)^\nu)^\nu = (a^\nu b)^\nu = a^\nu b^\nu. \tag{3.4}$$

To replace the forgotten addition of F, we define a new addition in $F^\times$ based on the order of $\mathcal{G}$:

$$a + b = \begin{cases} a & \text{if } a^\nu > b^\nu, \\ b & \text{if } a^\nu < b^\nu, \\ a^\nu & \text{if } a^\nu = b^\nu, \end{cases}$$

for all $a, b \in F^\times$. This actually gives addition rules for whole $F^\times \sqcup \mathcal{G}$, as can be seen in the proof of the following proposition.

Remark. The above kind of L-layered semiring†, where L consists of two elements, is called a *supertropical semiring*†, or more precisely a *supertropical semidomain*†

Proposition 3.32. *The disjoint union $F^\times \sqcup \mathcal{G}$, as given in Example 3.31, is an L-layered semidomain†.*

Proof. Clearly, L is a sorting semiring†. It is still required to show the following points:

(1) $(F^\times \sqcup \mathcal{G}, +)$ is an Abelian semigroup.

(2) $(F^\times \sqcup \mathcal{G}, \cdot)$ is a monoid with the unit element 1_F.

(3) Multiplication $(\cdot)$ distributes over addition $(+)$.

(4) Axioms (I) – (VII) in Definition 3.23 hold true.

(5) $F^\times \sqcup \mathcal{G}$ is commutative and cancellative.

This is done as follows.

(1) Let $a, b, a^\nu, b^\nu \in F^\times \sqcup \mathcal{G}$ (such that $a, b \in F^\times$ and $a^\nu, b^\nu \in \mathcal{G}$). Then $a + b \in \{a, b, a^\nu, b^\nu\} \subset F^\times \sqcup \mathcal{G}$, $a^\nu + b^\nu = a + a + b + b = (a + b)^\nu \in \mathcal{G} \subset F^\times \sqcup \mathcal{G}$, and similarly both $a + b^\nu = a + b + b \in \{a, b^\nu\} \subset F^\times \sqcup \mathcal{G}$, and $a^\nu + b = a + a + b \in \{a^\nu, b\} \subset F^\times \sqcup \mathcal{G}$. It is clear that addition is associative, as well as $(F^\times \sqcup \mathcal{G}, +)$ is Abelian. Hence, $(F^\times \sqcup \mathcal{G}, +)$ is an Abelian semigroup.

(2) Let $a, b, a^\nu, b^\nu \in F^\times \sqcup \mathcal{G}$ (such that $a, b \in F^\times$ and $a^\nu, b^\nu \in \mathcal{G}$). Then $ab \in F^\times \subset F^\times \sqcup \mathcal{G}$, and $ab^\nu = a^\nu b = a^\nu b^\nu = (ab)^\nu \in \mathcal{G} \subset F^\times \sqcup \mathcal{G}$. For the multiplicative neutral element, it holds that $1_F \in F^\times \subset F^\times \sqcup \mathcal{G}$, and the same element acts as a neutral element for the elements of $\mathcal{G}$. Namely,

$$1 \cdot a^\nu = (1 \cdot a)^\nu = a^\nu = (a \cdot 1)^\nu = a^\nu \cdot 1,$$

for all $a^\nu \in \mathcal{G}$ such that $a \in F^\times$. Associate property follows from that of both $F^\times$ and $\mathcal{G}$ and from 3.3. Hence, $(F^\times \sqcup \mathcal{G}, \cdot)$ is a monoid.

(3) To prove the distributive laws, we denote $a^{(\nu)}$ to mean either a or a^ν, for all $a, a^\nu \in F^\times \sqcup \mathcal{G}$. Let $a^{(\nu)}, b^{(\nu)}, c^{(\nu)} \in F^\times \sqcup \mathcal{G}$ such that $c^\nu < b^\nu$, implying $b^{(\nu)} + c^{(\nu)} = b^{(\nu)}$. The same assumption implies also $a^\nu c^\nu < a^\nu b^\nu$. This follows from Proposition 2.26, since $\mathcal{G}$ is totally ordered and cancellative (as a group). Therefore, it also holds that $ac^\nu < ab^\nu$, based on the multiplication rules in (3.3) and (3.4). This can be written as $(ac^\nu)^\nu < (ab^\nu)^\nu$, which implies $ac^\nu + ab^\nu = ab^\nu$, and thus, $a^\nu c + a^\nu b = a^\nu b$. On the other hand, $a^\nu c^\nu < a^\nu b^\nu$ can be written as $(ac)^\nu < (ab)^\nu$, which implies $ac + ab = ab$. Now

$$\begin{aligned}
&a^{(\nu)}(b^{(\nu)} + c^{(\nu)}) = a^{(\nu)}b^{(\nu)} = a^{(\nu)}b^{(\nu)} + a^{(\nu)}c^{(\nu)}, \\
&a^{(\nu)}(b^{(\nu)} + b^{(\nu)}) = a^{(\nu)}b^\nu = (a^{(\nu)}b)^\nu = a^{(\nu)}b + a^{(\nu)}b = a^{(\nu)}b^{(\nu)} + a^{(\nu)}b^{(\nu)}.
\end{aligned}$$

The other distributive law can be proved in a similar way.

(4) From the axioms, we have already proved axiom (I). When recalling the table for multiplication, given in Example 3.31, we have already proved axiom (II), too. Axioms (IV) and (VII) are also easy, they follow from the definition of the addition rules.

To prove axiom (V), we denote $a^{(\nu)}$ to mean either a or a^ν, for all $a, a^\nu \in F^\times \sqcup \mathcal{G}$. Suppose that $a^{(\nu)} + b^{(\nu)} = c^{(\nu)}$, for $a, b, c \in F^\times$ such that $a^\nu \neq b^\nu$. This equation implies $a^{(\nu)} + b^{(\nu)} + a^{(\nu)} + b^{(\nu)} = c^{(\nu)} + c^{(\nu)}$, which is the same as $a^\nu + b^\nu = c^\nu$.

To show axiom (III) to hold true, suppose first that $a, b \in R_1 = F^\times$. For clarity, we will below use the indices in the transition maps. We have the following cases:

$$\begin{aligned}
\nu_{1,1}(a)\nu_{1,1}(b) &= ab = \nu_{1,1}(ab) = \nu_{1\cdot 1,1\cdot 1}(ab),\\
\nu_{1,1}(a)\nu_{\infty,1}(b) &= ab^\nu = (ab)^\nu = \nu_{\infty,1}(ab) = \nu_{1\cdot\infty,1\cdot 1}(ab),\\
\nu_{\infty,1}(a)\nu_{1,1}(b) &= a^\nu b = (ab)^\nu = \nu_{\infty,1}(ab) = \nu_{\infty\cdot 1,1\cdot 1}(ab),\\
\nu_{\infty,1}(a)\nu_{\infty,1}(b) &= a^\nu b^\nu = (ab)^\nu = \nu_{\infty,1}(ab) = \nu_{\infty\cdot\infty,1\cdot 1}(ab).
\end{aligned}$$

If again $a \in R_1$, but $b \in R_\infty = \mathcal{G}$, when marking it as b^ν, we have the cases:

$$\begin{aligned}
\nu_{1,1}(a)\nu_{\infty,\infty}(b^\nu) &= ab^\nu = (ab)^\nu = \nu_{\infty,\infty}((ab)^\nu) = \nu_{1\cdot\infty,1\cdot\infty}(ab^\nu),\\
\nu_{\infty,1}(a)\nu_{\infty,\infty}(b^\nu) &= a^\nu b^\nu = (ab)^\nu = \nu_{\infty,\infty}((ab)^\nu) = \nu_{\infty\cdot\infty,1\cdot\infty}(ab^\nu).
\end{aligned}$$

The opposite case, where $a \in R_\infty$ and $b \in R_1$, is similar. If instead $a, b \in R_\infty$, we mark a^ν and b^ν and have the following case:

$$\nu_{\infty,\infty}(a^\nu)\nu_{\infty,\infty}(b^\nu) = a^\nu b^\nu = (ab)^\nu = \nu_{\infty,\infty}((ab)^\nu) = \nu_{\infty\cdot\infty,\infty\cdot\infty}(a^\nu b^\nu).$$

Hence, in each case, axiom (III) holds true.

We will next concentrate on axiom (VI), the supertropicality property. Let $a, b \in F^\times \sqcup \mathcal{G}$ such that $a \cong_\nu b$. This means that $a^\nu = b^\nu$, and thus, $a + b = a^\nu \in R_\infty$. Moreover, $a + b = a^\nu \cong_\nu a$. Suppose next that $a \in R_\infty$, and write it as $a = c^\nu$, for $c \in R_1$. Now, $a + b = a^\nu = (c^\nu)^\nu = c^\nu = a$.

(5) To show commutative property, suppose that $a, b \in F^\times$. Since F as a field is commutative, $ab = ba$, and thus, $a^\nu b^\nu = (ab)^\nu = (ba)^\nu = b^\nu a^\nu$. Together with the definition of the mixed multiplication in (3.3), this proves that $F^\times \sqcup \mathcal{G}$ is commutative. The cancellative property follows from that of both $F^\times$ and $\mathcal{G}$. Namely, F as a field is cancellative, and $\mathcal{G}$ is a (multiplicative) group, and thus, it includes the multiplicative inverses. The multiplication rules in (3.3) and (3.4) imply $F^\times \sqcup \mathcal{G}$ to be cancellative. We can also assume $F^\times \sqcup \mathcal{G}$ to be non-zero, since F is a field.

We have now proved $F^\times \sqcup \mathcal{G}$ to be an L-layered semidomain†. $\square$

Remark. Actually, ν satisfies the requirements (i) and (ii) of a valuation. Namely, in (3.4) we have already shown that $(ab)^\nu = a^\nu b^\nu$ for all $a, b \in F^\times$. Assume next that $b^\nu < a^\nu$. Then

$$\nu(a + b) = \nu(a) = \max\{\nu(a), \nu(b)\}.$$

If $a^\nu < b^\nu$, we can make the corresponding conclusion. If $a^\nu = b^\nu$, we obtain

$$\nu(a+b) = \nu(\nu(a)) = \nu(a) = \max\{\nu(a), \nu(b)\}.$$

Hence, in every case, $\nu(a+b) = \max\{\nu(a), \nu(b)\} \leq \max\{\nu(a), \nu(b)\}$. By extending the domain of ν from $F^\times$ to F and by requiring that $\nu(0_F) = -\infty$, we can make ν a valuation. Then we should also extend $F^\times \sqcup \mathcal{G}$ with the zero element $-\infty$.

Actually $\mathcal{G} \subset R$ is an ideal. Namely, if $a^\nu, b^\nu \in \mathcal{G}$ and $c^{(\nu)} \in R$, then

$$\begin{aligned} a^\nu + b^\nu &= a + a + b + b = (a+b)^\nu \in \mathcal{G}, \\ c^{(\nu)} a^\nu &= (ca)^\nu \in \mathcal{G}. \end{aligned}$$

Example 3.33. As a more concrete version of the previous example, consider the situation where again $L = \{1, \infty\}$, but now we take two copies of $\mathbb{R}$ as the disjoint sets, i.e. $R_1 = \mathbb{R} = R_\infty$. We define multiplication in a tropical way to be addition of $\mathbb{R}$. Addition is as defined in the previous example. Now again, there is no zero element, since we have not added $-\infty$.

Since we have two identical sets, the transition map ν is an identity map from one copy to the other, and we again mark the images of the transition map as a^ν for all $a \in \mathbb{R}$. The whole union can be marked as $\mathbb{R} \sqcup \mathbb{R}^\nu$ and called a *standard supertropical semiring†*.

3.5.3 Several uniform layers

Example 3.34. Let L be a totally ordered sorting semiring†, and $\mathcal{T}$ a cancellative totally ordered multiplicative monoid. Based on Proposition 3.9, $\mathcal{T}$ can be viewed as a bipotent semiring†, since it is a totally ordered monoid. We construct a Cartesian product $R := L \times \mathcal{T}$, but instead of marking the elements as (k, a), we use the notation ${}^{[k]}a$, where $k \in L, a \in \mathcal{T}$. Here a is called an *actual value* and k a *layer (value).*

Operations are given as follows. Multiplication is defined componentwise, as

$${}^{[k]}a \cdot {}^{[l]}b = {}^{[kl]}(ab),$$

and addition is based on the order of $\mathcal{T}$, as

$${}^{[k]}a + {}^{[l]}b = \begin{cases} {}^{[k]}a & \text{if } a > b, \\ {}^{[l]}b & \text{if } a < b, \\ {}^{[k+l]}a & \text{if } a = b, \end{cases}$$

for all $k, l \in L, a, b \in \mathcal{T}$. We denote

$$R_k := \{{}^{[k]}a \mid a \in \mathcal{T}\},$$

when each R_k is like a layer in an L-layered semiring†. We define the transition maps

$$\nu_{l,k} : R_k \to R_l \text{ such that } {}^{[k]}a \mapsto {}^{[l]}a,$$

where $k, l \in L$ such that $l \geq k$, and $a \in \mathcal{T}$. Clearly, such maps are bijective. Finally, we can assume the element ${}^{[1]}1_{\mathcal{T}}$ to be the unit element of whole R, which will be seen in the proof of the following proposition.

Remark. The above structure is called a *uniform L-layered semiring†* and marked as $\mathcal{R}(L, \mathcal{T})$. It differs from an L-layered semiring† such that L is required to be totally ordered and the transition maps to be bijective.

If moreover $\mathcal{T}$ is Abelian, then the structure is a semidomain†, as will be proved in the following proposition. In this case, it can be called a *uniform L-layered semidomain†*.

Proposition 3.35. *The uniform L-layered semiring†, $R := \mathcal{R}(L, \mathcal{T})$, as given in Example 3.34, is an L-layered semiring†. If moreover $\mathcal{T}$ is Abelian, then R is a semidomain†.*

Proof. It is required to show the following points:

(1) $(R, +)$ is an Abelian semigroup.

(2) $(R, \cdot)$ is a monoid with the unit element ${}^{[1]}1_{\mathcal{T}}$.

(3) Multiplication $(\cdot)$ distributes over addition $(+)$.

(4) Axioms (I) – (VII) in Definition 3.23 hold true.

(5) R is cancellative, and if $\mathcal{T}$ is Abelian, then R is commutative.

This is done as follows.

(1) If we take two elements of R with different actual values, it is clear that the sum of these elements is an element of R. In this case, Abelian property is straigthforward. Assume next that ${}^{[k]}a, {}^{[l]}a \in R$, for $k, l \in L$ and $a \in \mathcal{T}$. Then ${}^{[k]}a + {}^{[l]}a = {}^{[k+l]}a$. Because L is a semiring†, $k + l \in L$, and thus, ${}^{[k+l]}a \in R$. In this case, Abelian property follows from that of L.

Associative property is clear among three elements, all of which have different actual values or all of which have identical actual values. Thus, it is sufficient to prove this property among three elements, two of which share the same actual value and the third one has an actual value different from that of the others. The Abelian property, proved above, guarantees that the order of the elements does not play any role. Let ${}^{[k]}a, {}^{[l]}a, {}^{[m]}b \in R$, for $k, l, m \in L$ and $a, b \in \mathcal{T}$. If $a < b$, then

$$({}^{[k]}a + {}^{[l]}a) + {}^{[m]}b = {}^{[k+l]}a + {}^{[m]}b = {}^{[m]}b, \text{ and}$$
$${}^{[k]}a + ({}^{[l]}a + {}^{[m]}b) = {}^{[k]}a + {}^{[m]}b = {}^{[m]}b.$$

If $a > b$, then

$$\begin{aligned}({}^{[k]}a + {}^{[l]}a) + {}^{[m]}b &= {}^{[k+l]}a + {}^{[m]}b = {}^{[k+l]}a, \text{ and} \\ {}^{[k]}a + ({}^{[l]}a + {}^{[m]}b) &= {}^{[k]}a + {}^{[l]}a = {}^{[k+l]}a.\end{aligned}$$

Hence, $(R, +)$ is an Abelian semigroup.

(2) Let ${}^{[k]}a, {}^{[l]}b \in R$, for $k, l \in L$ and $a, b \in \mathcal{T}$. Now ${}^{[k]}a \cdot {}^{[l]}b = {}^{[kl]}(ab) \in R$, because L and $\mathcal{T}$ are semirings† implying that $kl \in L$ and $ab \in \mathcal{T}$.

Because L and $\mathcal{T}$ are semirings† with multiplicative neutral elements, 1 and $1_{\mathcal{T}}$, respectively, it is clear that

$${}^{[1]}1_{\mathcal{T}} \cdot {}^{[k]}a = {}^{[1\cdot k]}(1_{\mathcal{T}} \cdot a) = {}^{[k]}a = {}^{[k\cdot 1]}(a \cdot 1_{\mathcal{T}}) = {}^{[k]}a \cdot {}^{[1]}1_{\mathcal{T}},$$

for all ${}^{[k]}a \in R$, where $k \in L$ and $a \in \mathcal{T}$.

Associative property follows from that of both L and $\mathcal{T}$.

(3) Distributive laws can be proved as follows. Suppose that ${}^{[k]}a, {}^{[l]}b, {}^{[m]}c \in R$, for $k, l, m \in L$ and $a, b, c \in \mathcal{T}$. Assume first that $b < c$, which based on Proposition 2.26 implies $ab < ac$, since $\mathcal{T}$ is ordered and cancellative. In this case,

$$\begin{aligned}{}^{[k]}a \cdot ({}^{[l]}b + {}^{[m]}c) &= {}^{[k]}a \cdot {}^{[m]}c = {}^{[km]}(ac) = {}^{[kl]}(ab) + {}^{[km]}(ac) \\ &= {}^{[k]}a \cdot {}^{[l]}b + {}^{[k]}a \cdot {}^{[m]}c.\end{aligned}$$

The case, where $b > c$, is similar. Assume next that $b = c$, which implies $ab = ac$. Since the distributive law holds in L, we have

$$\begin{aligned}{}^{[k]}a \cdot ({}^{[l]}b + {}^{[m]}c) &= {}^{[k]}a \cdot {}^{[l+m]}b = {}^{[kl+km]}(ab) = {}^{[kl]}(ab) + {}^{[km]}(ab) \\ &= {}^{[k]}a \cdot {}^{[l]}b + {}^{[k]}a \cdot {}^{[m]}c.\end{aligned}$$

The other distributive law can proved in a very similar way.

(4) Axiom (I) has already been proved. Axiom (II) follows from the definition of multiplication, while axioms (IV) and (VII) follow from that of addition.

To prove axiom (III), suppose that $a \in R_k$ and $b \in R_l$, for $k, l \in L$. This means that $a = {}^{[k]}u$ and $b = {}^{[l]}v$, for $u, v \in \mathcal{T}$. Suppose that $m, m' \in L$ such that $m \geq k$ and $m' \geq l$. Then

$$\nu_{m,k}(a)\nu_{m',l}(b) = {}^{[m]}u^{[m']}v = {}^{[mm']}(uv) = \nu_{mm',kl}(ab),$$

where the last equation follows from axiom (II).

When proving supertropicality, we will first give another form for ν-equivalence. Suppose that $a \cong_\nu b$, where $a \in R_k, b \in R_l$, for $k, l \in L$. This ν-equivalence means that

$$\nu_{m,k}(a) = \nu_{m,l}(b),$$

for some $m \in L$ such that $m \geq k, l$. When denoting $a = {}^{[k]}u$ and $b = {}^{[l]}v$, for $u, v \in \mathcal{T}$, this is equivalent to ${}^{[m]}u = {}^{[m]}v$, which again is the same as $u = v$. In other words, two elements are ν-equivalent, exactly when their actual values are equal. It is now easy to see that axiom (VI) holds true.

Axiom (V) can be proved as follows. Suppose that $a \in R_k$, $b \in R_l$, and $c \in R_m$, where $k, l, m \in L$, such that $a + b = c$ and $a \not\cong_\nu b$. The equation can be written as ${}^{[k]}u + {}^{[l]}v = {}^{[m]}w$, for $u, v, w \in \mathcal{T}$. Recall the same interpretation for ν-equivalence as given when proving axiom (VI). Since $a \not\cong_\nu b$, we can assume, for example, that $a <_\nu b$, i.e. $u < v$. The above equation can now be written as ${}^{[k]}u = {}^{[m]}w$, and thus, $k = m$ and $u = w$. It is easy to see that the equation ${}^{[p]}u + {}^{[p]}v = {}^{[p]}w$, holds true for all $p \in L$.

(5) Cancellative property of R follows from that of $\mathcal{T}$ and from axiom (III). If $\mathcal{T}$ is Abelian, then clearly R is commutative. We can also assume R to be non-zero, since $\mathcal{T}$ is multiplicative.

□

Remark. The problem in addition arising in a general layered semiring†, as discussed in Example 3.28, does not occur in a uniform layered semiring†. Namely, based on point (4) in the above proof, we can see that distinct elements, sharing the same layer in a uniform layered semiring†, are not ν-equivalent with each other.

Example 3.36. Recall Example 3.33 on an L-layered semiring†, $\mathbb{R} \sqcup \mathbb{R}^\nu$. We used it as an example on two layers, but it serves also as an example of uniform layers, since both of those two layers are identical. Actually,

$$\mathbb{R} \sqcup \mathbb{R}^\nu \cong \mathcal{R}(L, \mathbb{T}^*),$$

where $L = \{1, \infty\}$.

We have already proved $\mathbb{R} \sqcup \mathbb{R}^\nu$ to be an L-layered semidomain† (see Proposition 3.32 and Example 3.33). The same holds true for $R = \mathcal{R}(L, \mathbb{T}^*)$. Namely, $\mathbb{T}^*$ is a totally ordered cancellative multiplicative monoid, and according to the tables for $L = \{1, \infty\}$, given in Example 3.31, we can see that L is a semiring† with the unit element 1. Hence, based on Proposition 3.35, R is an L-layered semiring†. Clearly, it is also an L-layered semidomain†.

To see that also the operations fit together, we define the map

$$\begin{aligned} \varphi : \mathbb{R} \sqcup \mathbb{R}^{\nu} &\to R, \\ a &\mapsto {}^{[1]}a, \\ a^{\nu} &\mapsto {}^{[\infty]}a, \end{aligned}$$

which can be easily proved to be an isomorphism.

We will finally return to the claim that $\mathbb{T}^*$ is not a good choice for a sorting semiring†.

Example 3.37. Let $R := \mathcal{R}(\mathbb{T}^*, \mathbb{T}^*)$ be a uniform $\mathbb{T}^*$-layered semiring†. We have earlier mentioned (in Example 3.22) that such a semiring† does not remove the idempotency property. Indeed, if ${}^{[k]}a \in R$, for $a, k \in \mathbb{T}^*$, then

$${}^{[k]}a \oplus {}^{[k]}a = {}^{[k \oplus k]}a = {}^{[k]}a.$$

Remark. Based on the above example, it is reasonable to assume that a sorting semiring† does not follow max-plus algebra.

3.6 Partially ordered semirings

Recall the connection between a totally ordered monoid and a bipotent semiring† in Proposition 3.9. We will next loosen the requirement of a total order to a partial one. Such an action is reasonable, since although a semiring† is totally ordered, a polynomial over the semiring† is only partially ordered. However, we will just consider the structure to be introduced as an example of a layered semiring†.

We start with the following definitions given in [21, p. 2] and [21, p. 4], respectively.

Definition 3.38. Let M be a (multiplicative) Abelian monoid and $S \subset M$ a subset. It is said that M is *cancellative with respect to* S, if $as = bs$ implies $a = b$, for all $a, b \in M$ and $s \in S$. Then S is called a *cancellative subset* of M.

Definition 3.39. A ν-semiring† is a quadruple $R := (R, \mathcal{T}, \mathcal{G}, \nu)$, where R is a semiring†, $\mathcal{T} \subset R$ is a multiplicative submonoid, $\mathcal{G} \subset R$ is a partially ordered semiring† ideal, together with a map $\nu : R \to \mathcal{G}$, satisfying $\nu^2 = \nu$ as well as the conditions:

$$a + b = \begin{cases} a, & \text{if } \nu(a) > \nu(b), \text{ i.e. } a >_{\nu} b, \\ \nu(a), & \text{if } \nu(a) = \nu(b), \text{ i.e. } a \cong_{\nu} b, \end{cases}$$

for all $a, b \in R$.

If $(R, \cdot)$ is commutative and cancellative with respect to $\mathcal{T}$, R is called a *ν-semidomain*†. If furthermore $\mathcal{T}$ is an Abelian group, R is called a *ν-semifield*†.

Example 3.40. A layered semiring† $R := F^\times \sqcup \mathcal{G}$, presented in Example 3.31, is a ν-semiring†. First, $F^\times$ is a multiplicative group, and thus, a multiplicative monoid. Second, $\mathcal{G}$ is totally ordered, and based on the remark after Proposition 3.32, it is an ideal of R. Finally, the map ν introduced in Example 3.31 satisfies the requirements given in Definition 3.39. Moreover, R is a ν-semifield†, since $F^\times$ is an Abelian group, and commutative and cancellative properties were proved in Proposition 3.32.

We will show that a layered semiring†, as given in Definition 3.23 can be seen as a ν-semiring†.

Proposition 3.41. *Let L be a sorting semiring† and $R := (R, L, (\nu_{m,l}))$ an L-layered semiring†. Then $R \sqcup R_\infty$, where*

$$R_\infty := R/\cong_\nu,$$

is a ν-semiring†.

Proof. The sorting semiring† of $R \sqcup R_\infty$ is $L \sqcup \{\infty\}$. Note that L may have one or several infinite elements in the first place, but ∞ is assumed to be greater than all the original elements of L, and thus, this element can act as the common upper bound for each pair of elements in $L \sqcup \{\infty\}$. Therefore R_∞ can be considered as a direct limit of the direct system consisting of the layers and the sort transition maps of R, in the same way as in [16, p. 11]. Moreover, we can now assume the sorting semiring† of $R \sqcup R_\infty$ to have at least two elements.

Since R is a layered semiring†, also $R \sqcup R_\infty$ is such, and from now on, we denote R for $R \sqcup R_\infty$, as well as, L for $L \sqcup \{\infty\}$. This layered semiring† consists of a family $(R_l)_{l\in L}$ of disjoint subsets $R_l \subset R$. The claim follows by choosing $\mathcal{T} := R_1$, $\mathcal{G} := R_{>1}$, and defining ν as follows:

$$\nu(a) = \nu_{\infty,k}(a),$$

for all $a \in R_k$ and $k \in L$.

Based on the remark after Definition 3.23, R_1 is a multiplicative monoid, and based on Proposition 3.29, $R_{>1}$ is an ideal of R. Moreover, based on Proposition 3.27, $(R, \leq_\nu)$ is totally ordered, and thus, also the subset $(R_{>1}, \leq_\nu)$ is totally ordered.

When setting the map ν as above, it clearly satisfies $\nu^2 = \nu$. The remaining task is to show that it satisfies also the given conditions. For this purpose, let $k, l \in L$ and $a \in R_k, b \in R_l$. If $a >_\nu b$, then $a + b = a$. This can be concluded in the same way as done in Example 3.28. If $a \cong_\nu b$, then based on supertropicality axiom, $a + b \in R_{k+l}$ and $a + b \cong_\nu a$. The latter condition means that $\nu(a + b) = \nu(a)$. The former one implies $a + b \in R_{>1}$, when it holds that $\nu(a+b) = a+b$. Namely, we can extend the map $\nu : R_1 \to R_{>1}$ to the map $\nu : R \to R_{>1}$ by assuming it to be an identity map for the elements of $R_{>1}$. Therefore the equations $\nu(a + b) = \nu(a)$ and $\nu(a + b) = a + b$ imply together that $a + b = \nu(a)$. □

Chapter 4

Layered tropical polynomials

4.1 Polynomials over a layered semiring

This chapter is devoted to the polynomials over a semiring†. We will consider the concepts introduced in previous chapter in the context of polynomials. Again, we will use a semiring† as a generalization of a semiring. Before discussing polynomials, we need the following definitions.

Definition 4.1. Let R be a layered semidomain†. It said that R is a *1-semifield*†, if $R_1 \subset R$ is an Abelian group.

If a 1-semifield† includes a zero element, it is called a *1-semifield*.

Remark. Recall that R_1 is a multiplicative monoid (as mentioned in the remark after Definition 3.23). If, moreover, R_1 is an Abelian group, all its elements have multiplicative inverses, except for the possible zero element. Hence, R_1 is a semifield†, meaning that R is a 1-semifield†.

Even if R_1 is an Abelian group, the other layers do not share this property, except for an infinite layer, denoted as R_∞. Namely, suppose that L is a sorting semiring†, and R an L-layered 1-semifield†. If $1 \neq k \in L$ is finite, and $a, b \in R_k$, then $ab \in R_{kk} \neq R_k$. This follows from axiom (II) in Definition 3.23.

However, R_∞ is an Abelian group, if R_1 is such. Namely, if $a_1 \in R_1$, then $(a_1)^{-1} \in R_1$. Furthermore, $1_\infty := \nu_{\infty,1}(1)$ is the neutral element in R_∞, and since each $a_\infty \in R_\infty$ can be written as $a_\infty = 1_\infty \cdot a_1$, we have

$$a_\infty \cdot \underbrace{1_\infty \cdot (a_1)^{-1}}_{\in R_\infty} = 1_\infty \cdot a_1 \cdot 1_\infty \cdot (a_1)^{-1} = 1_\infty,$$

which proves that each $a_\infty \in R_\infty$ has a multiplicative inverse in R_∞.

As a semidomain†, 1-semifield† is cancellative, but the following lemma shows how this realizes in respect to ν-equivalence.

Lemma 4.2. *A 1-semifield† is cancellative (in respect to ν-equivalence).*

Proof. Let K be a 1-semifield†, $a, b, c \in K$, and $c_1 \in K_1$ such that $c_1 \cong_\nu c$. There always exists c_1, if we assume sort transition maps to be surjective (as mentioned in the remark after Definition 3.23). Now $c_1 \cong_\nu c$ implies both $ac_1 \cong_\nu ac$ and $bc_1 \cong_\nu bc$. If $ac \cong_\nu bc$, then transitive property of ν-equivalence implies $ac_1 \cong_\nu bc_1$. Since K is a 1-semifield† and $c_1 \in K_1$, it holds that $(c_1)^{-1} \in K_1 \subset K$. Multiplying both sides of the previous ν-equivalence with $(c_1)^{-1}$ gives $a \cong_\nu b$. □

Most often a 1-semifield† is not a semifield†.

Lemma 4.3. *Let L be a sorting semiring†, and R an L-layered 1-semifield†. Then R is a semifield†, if and only if L is such.*

Proof. Suppose first that L is not a semifield†. Then there exists (a non-zero) $k \in L$ without a multiplicative inverse. If $a_k \in R_k$, then a_k has no multiplicative inverse either, since to hold, the equation

$$a_k \cdot b_l = 1_R \in R_1,$$

for $l \in L$ and $b \in R_l$, would require l to be a multiplicative inverse of k.

Suppose then that L is a semifield†. Let a_k be an arbitrary element of R, for $k \in L$. Now $k^{-1} \in L$ (when assuming k to be non-zero), and further $1_{k^{-1}} \in R$. By assuming the sort transitions maps to be surjective (as mentioned in the remark after Definition 3.23), there exists $a_1 \in R_1$ such that $a_1 \cong_\nu a_k$. Since R is a 1-semifield†, we have $(a_1)^{-1} \in R$. Therefore

$$a_k \cdot 1_{k^{-1}} \cdot (a_1)^{-1} = a_1 \cdot (a_1)^{-1} = 1_R,$$

which proves the claim. □

Remark. Most often, we have $L = L_{\geq 1}$ containing several elements, when an L-layered 1-semifield† is not a semifield†.

Example 4.4. Let $L := \{1\}$ be a sorting semiring†. Now $\mathcal{R}(L, \mathbb{T}^*)$ is a 1-semifield†. It is also a semifield†, and the same holds true for L.

Definition 4.5. Let R be a layered semiring†. It is said that R is *1-divisibly closed*, if for every $b \in R_1$ and $m \in \mathbb{N}^*$, there is $a \in R_1$, for which $a^m = b$.

Example 4.6. Let $L = \{1, \infty\}$. An L-layered semiring† $\mathbb{Q} \sqcup \mathbb{Q}^\nu := \mathcal{R}(L, \mathbb{Q})$ (compare to $\mathbb{R} \sqcup \mathbb{R}^\nu$ in Example 3.33) is 1-divisibly closed. Namely, here the multiplication is the original addition of $\mathbb{Q}$, and thus, the condition of the 1-divisibly closed semiring† becomes in the form: $ma = b$. This holds true with rational numbers, since $a = b/m \in \mathbb{Q}$.

Remark. If a semiring† is 1-divisibly closed, it is possible to take roots such that the result of this action belongs to the semiring†. Such actions are needed when calculating the roots of a polynomial. However, if we just declare a polynomial without any need to calculate its roots, we will not require the underlying semiring† to be 1-divisibly closed.

Now we are ready to start with polynomials. As shown in Example 2.14, tropical mathematics in general has the property that different polynomials may represent the same polynomial function. This can be seen in the following example.

Example 4.7. Consider the polynomials

$$F = X^3 \oplus X^2 \oplus 4X \oplus 3 \quad \text{and} \quad G = X^3 \oplus 4X \oplus 3,$$

both of which are polynomials over $\mathbb{T}$. We can define the corresponding functions:

$$f : x \mapsto F(x) \quad \text{and} \quad g : x \mapsto G(x),$$

both of which are maps $\mathbb{T} \to \mathbb{T}$. Clearly, F and G are distinct polynomials, but f and g are the same piecewise linear functions, as discussed in Example 2.14.

Remark. The previous example reveals a difference between tropical algebra and usual commutative algebra. Usually, if polynomials are taken over an infinite field, there is a one-to-one correspondence between the polynomials and the polynomial functions. Such a property does not hold in the tropical world.

We have already introduced real tropical polynomials over $\mathbb{T}$ in Definition 2.11. Since we now operate at a more general level than done in Chapter 2, we give a new definition for tropical polynomials.

Definition 4.8. Let R be a semiring†, and $X_1, \ldots, X_n$ indeterminants. We define

$$R[X_1, \ldots, X_n] := \{\sum_{\boldsymbol{i} \in \mathbb{N}^n} a_{\boldsymbol{i}} \boldsymbol{X}^{\boldsymbol{i}} \mid a_{\boldsymbol{i}} \in R, \text{almost all } a_{\boldsymbol{i}} = 0_R\},$$

where $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, and $\boldsymbol{X}^{\boldsymbol{i}} = X_1^{i_1} \cdots X_n^{i_n}$.

Each element of the above set is called a *tropical polynomial (over R)*. If F is a tropical polynomial, each term $a_{\boldsymbol{i}} \boldsymbol{X}^{\boldsymbol{i}}$ is called a *monomial* of F. If a polynomial is a sum of two monomials, it is called a *binomial*.

If the number of indeterminants (n) is negligible, we can write $R[\boldsymbol{X}]$ in the place of $R[X_1, \ldots, X_n]$.

Remark. The above set is allowed to contain such polynomials that represent the same function. For instance, if F and G are polynomials introduced in Example 4.7, then $F, G \in \mathbb{T}[X]$ (such that $F \neq G$).

Definition 4.9. Let R be a semiring† and S a set. We denote

$$\mathrm{Fun}(S, R) := \{f \mid f : S \to R \text{ is a function}\},$$

the set of functions from S to R.

Remark. Most typically $S \subset R^n$. As in usual algebra, $\mathrm{Fun}(S, R)$ can be made into a semiring† by defining addition and multiplication pointwise:

$$(f+g)(a) = f(a) + g(a) \quad \text{and} \quad (fg)(a) = f(a)g(a),$$

for all $f, g \in \mathrm{Fun}(S, R)$ and $a \in S$.

We will mainly concentrate on polynomial functions, but we need a more general definition to allow piecewise-defined functions, since tropical polynomial functions are actually such (the sub-functions of which are polynomial functions). It does not matter that other kinds of functions are not restricted away from $\mathrm{Fun}(S, R)$, since the results presented for $\mathrm{Fun}(S, R)$ are general enough to hold true for any functions.

Definition 4.10. Let R be a semiring† and S a set. There is a natural homomorphism

$$\begin{aligned} \varphi : R[\boldsymbol{X}] &\to \mathrm{Fun}(S, R) \\ F &\mapsto (a \mapsto F(a)), \end{aligned}$$

the image of which is denoted as $\mathrm{Pol}(S, R)$.

The above homomorphism is not necessarily injective. Therefore, we have an equivalence relation

$$F \sim G \quad \iff \quad \varphi(F) = \varphi(G).$$

We define $R\lfloor \boldsymbol{X} \rfloor := R[\boldsymbol{X}]/\sim$.

Remark. The above denotation is inspired by [19, pp. 14–15], but we mark $R\lfloor \boldsymbol{X} \rfloor$ and $R[\boldsymbol{X}]$ in a way opposite to that in [19]. In other words, we use the notation $R[\boldsymbol{X}]$ for proper polynomials, i.e. for the formal sums of monomials, as usually. Instead $R\lfloor \boldsymbol{X} \rfloor$ consists of the equivalence classes, where polynomials are equivalent whenever they determine the same polynomial function.

When we discuss polynomials, i.e. the elements of $R[\boldsymbol{X}]$, we typically use upper case letters such as F and G, instead of lower case letters such as f and g, which are reserved for polynomial functions.

4.2 Dominance of polynomials

This section concentrates more formally on the concepts discussed in the previous section. In the following definition, we return to the relations given in Definitions 3.23 and 3.25.

Definition 4.11. Let R be a layered semiring†, $S \subset R^n$ a subset, $f, g \in \mathrm{Fun}(S, R)$, and $a \in S$. It is said that

(i) f *dominates* g *at point* a, if $f(a) \geq_\nu g(a)$,

(ii) f *strictly dominates* g *at point* a, if $f(a) >_\nu g(a)$,

(iii) f *dominates* g, if $f(a) \geq_\nu g(a)$ for all $a \in S$, when we denote $f \geq_\nu g$,

(iv) f *strictly dominates* g, if $f(a) >_\nu g(a)$ for all $a \in S$, when we denote $f >_\nu g$,

(v) f and g are *ν-equivalent*, if it holds both $f \geq_\nu g$ and $g \geq_\nu f$, which imply $f \cong_\nu g$. (The implication follows from the proof of Proposition 3.27, based on which relation $\geq_\nu$ is antisymmetric.)

Remark. If f strictly dominates g, then $f + g = f$. If f dominates g, then $f + g \cong_\nu f$.

If $F \in R[X_1, \dots, X_n]$ and f is defined as

$$f : S \to R, \quad a \mapsto F(a),$$

then $f(a) = F(a)$, for all $a \in S$. Therefore, we can speak on dominance in the context of polynomials, in the same way as spoken in the previous definition in the context of functions.

Definition 4.11 compares two polynomials. A typical situation is to compare a polynomial to one of its monomials. The following lemma considers such an important special case of the previous definition.

Lemma 4.12. *Let R be a layered semiring†, $F \in R[X_1, \dots, X_n]$, and $a \in R^n$. If F_i is a monomial in F, then F_i dominates F at a, exactly when $F(a) \cong_\nu F_i(a)$. In other words,*

$$F_i(a) \geq_\nu F(a) \quad \iff \quad F_i(a) \cong_\nu F(a).$$

Proof. Suppose that $F_i(a) \geq_\nu F(a)$. Since the value of a single monomial can never be greater than the value of the whole polynomial, we always have $F(a) \geq_\nu F_i(a)$. Based on antisymmetry, $F_i(a) \cong_\nu F(a)$.

The other direction is obvious, as breafly shown at the very end of the remark after Definition 3.25. □

In the following proposition, we will apply ν-equivalence to a tuple, which is done componentwise to each element of the tuple, as follows. If R is a semiring†, and $a, b \in R^n$, i.e. $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, where $a_1, \dots, a_n, b_1, \dots, b_n \in R$ for all $i \in \{1, \dots, n\}$, then

$$a \cong_\nu b \quad \iff \quad a_i \cong_\nu b_i, \text{ for all } i \in \{1, \dots, n\}.$$

Proposition 4.13. *Let R be a layered semiring† and $F \in R[X_1, \dots, X_n]$. If $a, b \in R^n$ such that $a \cong_\nu b$, then $F(a) \cong_\nu F(b)$.*

Proof. Let F_i and F_j be monomials in F such that F_i dominates F at a and F_j dominates F at b. (Possibly F_i and F_j are the same monomials.) Based on Lemma 4.12, this means that

$$F_i(a) \cong_\nu F(a) \quad \text{and} \quad F_j(b) \cong_\nu F(b).$$

Since $a \cong_\nu b$, we have $F_i(a) \cong_\nu F_i(b)$. This follows from Proposition 3.26, which proves the relation $\cong_\nu$ to respect multiplication. The same equivalence holds for the other monomial, i.e. $F_j(a) \cong_\nu F_j(b)$. By putting all these things together, we have

$$\begin{aligned} F(a) \cong_\nu F_i(a) \cong_\nu F_i(b) \leq_\nu F(b), \\ F(b) \cong_\nu F_j(b) \cong_\nu F_j(a) \leq_\nu F(a), \end{aligned}$$

where the inequations follow from the fact that a single monomial cannot reach a greater value than the whole polynomial. The inequations imply together that $F(a) \cong_\nu F(b)$. $\square$

The following lemma considers the situation, where the value of single monomial is equal to the value of the whole polynomial at a certain point.

Lemma 4.14. *Let L be a sorting semiring†, R an L-layered semiring†, $a \in R^n$, and $F \in R[X_1, \ldots, X_n]$. Write F as a sum of monomials, i.e. $F = \sum_{i=1}^r F_i$, and suppose that F_i is a monomial in F ($i \in \{1, \ldots, r\}$). If L comprises no infinite elements, then*

$$F(a) = F_i(a) \qquad \Longleftrightarrow \qquad F_i \text{ is the only monomial dominating at } a.$$

Proof. If F_i is a single monomial dominating at a, then clearly $F(a) = F_i(a)$. Namely, since no other monomial dominates at a, it holds $F_j(a) \ngeq_\nu F(a)$, for all $j \neq i$. Therefore $F(a) >_\nu F_j(a)$, for all $j \neq i$.

To prove the other direction, suppose that $k_i \in L$ for all $i \in \{1, \ldots, r\}$, and $F_i(a) \in R_{k_i}$ for all $i \in \{1, \ldots, r\}$. Based on axiom (VI) in Definition 3.23, we can conclude that the layer of $F(a)$ is the sum of the layers of those monomials in F that dominate at a, i.e.

$$\sum_{F(a) \cong_\nu F_i(a)} k_i.$$

By denoting the above sum as k, it holds that $F(a) \in R_k$.

If F_i is not a single monomial dominating at a, then either F_i does not dominate at all, or F_i dominates together with another monomial. In the latter case, $F(a) \cong_\nu F_i(a)$, but if L lacks infinite elements, the layers of $F(a)$ and $F_i(a)$ differ from each other, and thus, $F(a) \neq F_i(a)$. In the former case, $F_i(a) \ngeq_\nu F(a)$, which implies $F(a) \ncong_\nu F_i(a)$ and further $F(a) \neq F_i(a)$. $\square$

Remark. Based on the above proof, the implication

$$F_i \text{ is the only monomial dominating at } a \quad \Longrightarrow \quad F(a) = F_i(a)$$

holds true in general, i.e. even if there are infinite elements in the sorting semiring†.

Recall Definition 2.13 for an essential monomial of a real tropical polynomial. We will next give a more general definition in the layered context. The following definition is derived from the definitions given in [15, p. 28] (Definition 5.5), [19, p. 16] (Definition 4.9 and Remark 4.10), and [20, p. 5] (Definition 3.10).

Definition 4.15. Let R be a layered semiring†, $F \in R[X_1, \dots, X_n]$, and $a \in R^n$. If F consists of a single monomial, we say that the monomial is *essential (in F) (at any point $a \in R^n$)* and not *inessential (in F) (at any point $a \in R^n$).*

Otherwise write $F = \sum_{i=1}^r F_i$, as a sum of monomials, let $j \in \{1, \dots, r\}$, and denote

$$H_j := \sum_{i=1, i \neq j}^{r} F_i,$$

the polynomial without monomial F_j.

A monomial F_j is *inessential (in F) at a*, if $F_j(a) <_\nu H_j(a)$. A monomial F_j is *inessential (in F)*, if it is inessential at all $a \in R^n$.

A monomial F_j is *essential (in F) at a*, if $F_j(a) >_\nu H_j(a)$. A monomial F_j is *essential (in F)*, if it is essential at some $a \in R^n$.

Remark. It is possible that a monomial F_j is neither essential nor inessential (at a). This happens, if $F_j(a) \geq_\nu H_j(a)$ for some $a \in R^n$, and $F_j(a) \leq_\nu H_j(a)$ for all $a \in R^n$. In other words, $F_j(a) \cong_\nu H_j(a)$ for some point $a \in R^n$ and $F_j(b) \leq_\nu H_j(b)$ for all other points $b \in R^n$.

There is an equivalent way to describe essential and inessential monomials [15], if the sorting semiring† of a layered semiring† has no infinite elements. The following lemma proves the equivalence between these descriptions.

Lemma 4.16. *Let L be a sorting semiring†, R an L-layered semiring†, $a \in R^n$, and $F \in R[X_1, \dots, X_n]$ a polynomial with at least two monomials. Write F as a sum of monomials, i.e. $F = \sum_{i=1}^r F_i$, and denote*

$$H_j = \sum_{i=1, i \neq j}^{r} F_i,$$

the polynomial without monomial F_j. If L comprises no infinite elements, then

(i) $F_j(a) <_\nu H_j(a)$, exactly when $F(a) = H_j(a)$,

(ii) $F_j(a) >_\nu H_j(a)$*, exactly when* $F(a) = F_j(a)$*.*

Proof. (i) Suppose that $F(a) = H_j(a)$, when further

$$H_j(a) = F(a) = (F_j + H_j)(a) = F_j(a) + H_j(a).$$

Therefore $F_j(a) \leq_\nu H_j(a)$, but if it was $F_j(a) \cong_\nu H_j(a)$ in the case where L has no infinite elements, then the above equation could not hold true. Hence $F_j(a) <_\nu H_j(a)$.

Suppose that $F_j(a) <_\nu H_j(a)$. This means that F_j does not dominate F at a, and thus, $F_j(a)$ has no effect, when calculating $F(a)$. Therefore $F(a) = H_j(a)$.

(ii) Note that the condition $F_j(a) >_\nu H_j(a)$ holds true, exactly when F_j is the only monomial in F dominating at a. If L comprises no infinite elements, the claim follows from Lemma 4.14.

□

Remark. An alternative way to define essential and inessential monomials is to replace the inequations in Definition 4.15 with the equations introduced in Lemma 4.16. This has been done in [15, p. 28] (Definition 5.5). Therefore the previous lemma proves that Definition 4.15 is equivalent to that given in [15], in the case where the sorting semiring† has no infinite elements. We prefer Definition 4.15, since it works well also with infinite elements in the sorting semiring†. (This will be discussed in a more concrete way in the remark after forth-coming Example 4.18.)

The previous lemma assumes the polynomial to have at least two monomials. The correspondence between the definitions holds true also for monomials, if R is a proper semiring†. Namely, if F is a monomial, then it is essential and not essential (based on Definition 4.15). Trivially also $F(a) = F_j(a)$ holds true, and we can assume the condition $F(a) = H_j(a)$ to not hold true, since there is no H_j, if $F = F_j$. However, if R is a semiring and F is a monomial, then $H_j = 0$, but now the condition $F(a) = H_j(a)$ can be true, although $F = F_j$ is not inessential. Such a situation occurs, if $a = 0$.

Example 4.17. Consider the polynomial

$$F = X^3 \oplus X^2 \oplus 4X \oplus 3 \in \mathbb{T}^*[X],$$

and denote

$$F_2 = X^2 \quad \text{and} \quad H_2 = X^3 \oplus 4X \oplus 3.$$

Now, F_2 is inessential in F. Namely, otherwise there should exist $a \in \mathbb{T}^*$ such that F_2 would dominate H_2 at a. This would mean that

$$2a \geq 3a \quad \text{and} \quad 2a \geq a + 4 \quad \text{and} \quad 2a \geq 3,$$

which is impossible, since the first two inequations cannot hold true at the same time.

On the other hand, by setting

$$F_1 = 4X \quad \text{and} \quad H_1 = X^3 \oplus X^2 \oplus 3,$$

we can conclude that F_1 is essential in F. Namely, there exists $0 \in \mathbb{T}$ such that

$$\begin{aligned} F_1(0) &= 4 \odot 0 = 4 + 0 = 4, \\ H_1(0) &= 3 \cdot 0 \oplus 2 \cdot 0 \oplus 3 = 3. \end{aligned}$$

Hence $F_1(0) > H_1(0)$, which means that F_1 is essential in F. In the corresponding way, with suitable values, we can conclude that also the monomials $F_0 = 3$ and $F_3 = X^3$ are essential in F.

It was mentioned in the remark after Definition 4.15 that a monomial can be neither essential nor inessential. The following example shows in a concrete way how this realizes.

Example 4.18. Consider the uniform layered semiring†, $R = \mathcal{R}(\mathbb{N}^*, \mathbb{T}^*)$ and its polynomial

$$F = X^2 \oplus {}^{[1]}1X \oplus {}^{[1]}2 \in R[X].$$

Denote $F_1 = {}^{[1]}1X$, and $H_1 = X^2 \oplus {}^{[1]}2$. Now,

$$\begin{aligned} F({}^{[1]}1) &= ({}^{[1]}1 \odot {}^{[1]}1) \oplus ({}^{[1]}1 \odot {}^{[1]}1) \oplus {}^{[1]}2 = {}^{[1]}2 \oplus {}^{[1]}2 \oplus {}^{[1]}2 = {}^{[3]}2, \\ H_1({}^{[1]}1) &= ({}^{[1]}1 \odot {}^{[1]}1) \oplus {}^{[1]}2 = {}^{[1]}2 \oplus {}^{[1]}2 = {}^{[2]}2, \\ F_1({}^{[1]}1) &= {}^{[1]}1 \odot {}^{[1]}1 = {}^{[1]}2. \end{aligned}$$

Since $F_1({}^{[1]}1) \cong_\nu H_1({}^{[1]}1)$, F_1 is not inessential in F.

If F_1 was essential in F, there would exist $a \in R$ such that $H_1(a) <_\nu F_1(a)$. Then it should hold that

$$a \odot a <_\nu {}^{[1]}1 \odot a \quad \text{and} \quad {}^{[1]}2 <_\nu {}^{[1]}1 \odot a,$$

and thus,

$$a <_\nu {}^{[1]}1 \quad \text{and} \quad {}^{[1]}1 <_\nu a.$$

Since this is impossible, F_1 is not essential in F.

Remark. Suppose that F in the example above is a polynomial over $R := \mathcal{R}(\{1\}, \mathbb{T}^*)$, when we just ignore the layer values. (Note that now the layer 1 is infinite, as shown in Example 3.30.) By following Definition 4.15, we can conclude that monomial $F_1 = 1X$ is not essential nor inessential, in the same way as above.

However, if the terms ”essential” and ”inessential” were defined according to [15], i.e. by using the equations introduced in Lemma 4.16, then the monomial $F_1 = 1X$ would be both essential and inessential. It is essential, since $F_1(1) = 2 = F(1)$.

Moreover, it holds that $F(a) = H_1(a)$ for all $a \in R$. To see this, suppose against the claim that $F(a) \neq H_1(a)$ for some $a \in R^n$. Therefore

$$H_1(a) \neq F(a) = (H_1 + F_1)(a) = H_1(a) + F_1(a).$$

Since $L = \{1\}$, we can conclude that $F_1(a) >_\nu H_1(a)$. Then it should hold that

$$1 + a > 2a \quad \text{and} \quad 1 + a > 2,$$

which imply $a < 1$ and $a > 1$, a contradiction. Hence, $F(a) = H_1(a)$ for all $a \in R$, which according to [15] means that F_1 is inessential.

Geometrically, it is easy to see which polynomial is the best representative of those representing the same function. The following definition gives a more formal criteria for the representative polynomial.

Definition 4.19. Let R be a layered semiring† and $F \in R[X_1, \ldots, X_n]$, written as $F = \sum_{i=1}^{r} F_i$. A *decomposition of* F is the sum taken over those monomials F_i that are not inessential. Moreover, we assume that the sum does not include such pairs of monomials F_i and F_j $(i \neq j)$ that either are the same or only differ from each other by their coefficient. Instead, we require the addition between such kind of monomials to be performed, i.e. $F_i + F_j$ is considered as a monomial.

4.3 Roots of polynomials

This section pays attention to the roots of polynomials. We have already discussed the corner roots of real tropical polynomials in Definition 2.15 and the remark after it. More generally, we will next give a new definition in the layered tropical context, taken from [20, p. 7] (Definition 4.2), which is below decomposed into two defintions.

Definition 4.20. Let R be a layered semiring†, $F \in R[X_1, \ldots, X_n]$, written as $F = \sum_{i=1}^{r} F_i$, and $a \in R^n$. The *corner support of* F *at* a is the set

$$\operatorname{csupp}_a(F) := \{F_i \text{ is a monomial in } F \mid F(a) \cong_\nu F_i(a)\}.$$

The *order* of $\operatorname{csupp}_a(F)$ means the number of elements in this set, denoted as $|\operatorname{csupp}_a(F)|$.

Remark. In other words, $\operatorname{csupp}_a(F)$ is the set of monomials in F dominating at a, while $|\operatorname{csupp}_a(F)|$ gives the number of monomials in F dominating at a.

If $F_i \in \operatorname{csupp}_a(F)$, then F_i is not inessential. This is clear if F is a monomial. Otherwise, if $F(a) \cong_\nu F_i(a)$, then F_i is a dominating monomial in F at a, and thus, $F_i(a) \geq_\nu H_i(a)$, where H_i is the sum taken over the other monomials of F except for F_i. Therefore, if $G \in R[X_1, \dots, X_n]$ is a decomposition of F, then $\mathrm{csupp}_a(G) = \mathrm{csupp}_a(F)$.

Definition 4.21. Let R be a layered semiring†, $S \subset R^n$ a subset, and $F \in R[X_1, \dots, X_n]$. The *corner locus of F (with respect to S)* is the set

$$\mathcal{Z}_{\mathrm{corn}}(F; S) := \{a \in S \mid |\,\mathrm{csupp}_a(F)| \geq 2\}.$$

If $S = R^n$, we write $\mathcal{Z}_{\mathrm{corn}}(F)$ instead of $\mathcal{Z}_{\mathrm{corn}}(F; R^n)$.

The elements of a corner locus are called *corner roots*.

Remark. In other words, a is a corner root of F, exactly when

$$F(a) \cong_\nu F_i(a) \cong_\nu F_j(a),$$

for some distinct monomials F_i and F_j in F, i.e. F has at least two monomials dominating at a. Namely, $a \in \mathcal{Z}_{\mathrm{corn}}(F)$, exactly when $|\,\mathrm{csupp}_a(F)| \geq 2$. This can be formulated as $F_i, F_j \in \mathrm{csupp}_a(F)$, where F_i and F_j are distinct monomials of F. This is equivalent to $F(a) \cong_\nu F_i(a)$ and $F(a) \cong_\nu F_j(a)$.

Clearly, layered tropical corner roots correspond to real tropical corner roots. They can be calculated in the same way, as shown in the following example.

Example 4.22. Consider the usual tropical situation, and suppose that

$$F = 2X^3 \oplus 7 \in \mathbb{T}[X].$$

Since $\mathbb{T}$ is 1-divisibly closed, we can calculate the roots in a normal way. Now F consists of two monomials, and the corner root can be found at the point where the monomials reach equivalent values. In usual tropical terminology, equivalence means equality, and we actually try to find the point where the corresponding lines cut each other. Hence we solve the equation $2 + 3X = 7$, and yield $X = \frac{5}{3}$.

As a layered tropical version with several layers, consider $R := \mathcal{R}(\mathbb{N}^*, \mathbb{T}^*)$, a uniform layered semiring†, which is also 1-divisibly closed. Suppose that $k, l \in \mathbb{N}^*$, and

$$F = {}^{[k]}2X^3 \oplus {}^{[l]}7 \in R[X].$$

To find the corner roots, we search for the points, where at least two monomials have ν-equivalent values. In the place of the equation above, we solve the ν-equivalence

$$\begin{aligned} {}^{[k]}2X^3 &\cong_\nu {}^{[l]}7 \quad \| \odot {}^{[1]}(-2) \\ {}^{[k]}0X^3 &\cong_\nu {}^{[l]}5 \cong_\nu {}^{[1]}5, \end{aligned}$$

and thus, $X^3 \cong_\nu {}^{[1]}5$. Since R is 1-divisibly closed, we can take the cube root. As a result, the actual value of X is $\frac{5}{3}$ and any layer value goes. All such points are corner roots, since both the monomials reach ν-equivalent values at these points.

Corner roots satisfy the following property in the case of several layers.

Lemma 4.23. *Let L be a sorting semiring† such that $|L| > 1$, R an L-layered semiring†, and $F \in R[X_1, \dots, X_n]$. If $a \in R^n$ is a corner root of F, then*

$$F(a) \in R_{>1}.$$

Proof. If a is a corner root of F, then $F(a) \cong_\nu F_j(a) \cong_\nu F_k(a)$, where F_j and F_k are distinct monomials in F. Write

$$F(a) = \sum_{i=1}^{r} F_i(a) = F_j(a) + F_k(a) + \sum_{i=1, i \neq j, i \neq k}^{r} F_i(a),$$

and suppose that $F_j(a) \in R_p$ and $F_k(a) \in R_q$, for $p, q \in L$. Now, monomials F_j and F_k are dominating ones (perhaps among others) in F at a. Based on the proof of Lemma 4.14, the layer of $F(a)$ is the sum taken over the layers of those monomials that dominate at a. Therefore $F(a) \in R_{\geq p+q}$. Since L has at least two elements, $p + q > 1$, and thus, $F(a) \in R_{>1}$. □

Remark. Based on the remark after Definition 3.23, the elements of $R_{>1}$ are called ghost elements or ghosts. Therefore the value of a polynomial at a corner root is a ghost.

In usual algebra, a root is defined to be a point, where the value of a polynomial is zero. In layered tropical mathematics, ghosts play the role of the missing zero. Besides the corner roots, there exist also other points, where the value of a polynomial is a ghost. The following definition introduces such points.

Definition 4.24. Let L be a sorting semiring†, R an L-layered semiring†, $S \subset R^n$ a subset, and $F = \sum_{i=1}^{r} F_i \in R[X_1, \dots, X_n]$. The *cluster locus of F (with respect to S)* is the set

$$\mathcal{Z}_{\text{clus}}(F; S) := \{a \in S \mid F_i(a) \in R_{>1} \text{ and } F(a) \not\cong_\nu F_j(a) \text{ for all } j \neq i\}.$$

If $S = R^n$, we write $\mathcal{Z}_{\text{clus}}(F)$ instead of $\mathcal{Z}_{\text{clus}}(F; R^n)$.

The elements of a cluster locus are called *cluster roots*.

Remark. Based on the latter condition of the above set, cluster roots are such points where the value of a polynomial is determined by a single monomial. Namely, based on Lemma 4.12, the condition $F(a) \not\cong_\nu F_j(a)$ for all $j \neq i$ means that none of the monomials F_j dominates F at a. Since there always has to be at least one monomial dominating at each point, the monomial dominating at a must be F_i. Therefore F_i is the only monomial determining the value of F at a cluster root.

Furthermore, the remark after Lemma 4.14 implies $F(a) = F_i(a)$. However, the opposite implication does not hold in general. Based on Lemma

4.14, it holds true, if L has no infinite elements. If the layer of $F_i(a)$ is infinite, it is possible that the equation $F(a) = F_i(a)$ holds, even if there were several monomials dominating at a.

Based on the first condition of the above set, the value of this single dominating monomial must be a ghost. Such a situation occurs, if the coefficient of the monomial is a ghost element, or otherwise if the root itself is a ghost element.

The above definition is formulated based on [15, p. 31]. However, another definition for a cluster root gives the following condition [18, pp. 28–29] (Definition 7.22):

$$F(a) = F_i(a) \in R_{>1},$$

which does not work well at infinite layers, as just discussed. If the layer of $F_i(a)$ is infinite, the above equation can hold even if a is a corner root. Since we wish to keep cluster locus and corner locus disjoint, we prefer the condition given in Definition 4.24.

As a conclusion, cluster locus consists of those points, where a single monomial in a polynomial dominates, but the value of the polynomial is nevertheless a ghost.

Example 4.25. Let L be a sorting semiring†, R an L-layered semiring†, and $F \in R[X_1, \ldots, X_n]$. If $L = \{1\}$, then $\mathscr{Z}_{\text{clus}}(F) = \emptyset$. Namely the condition $F_i(a) \in R_{>1}$ does not hold true for any monomial F_i in F nor for any value $a \in R$.

Example 4.26. Consider the uniform layered semiring† $R = (\mathbb{N}^*, \mathbb{T}^*)$ and its polynomial

$$F = X^3 \oplus {}^{[2]}4X \oplus {}^{[1]}3 \in R[X].$$

Now, 0 is a cluster root, since ${}^{[2]}4X$ is a single monomial dominating at 0, and ${}^{[2]}4X(0) = {}^{[2]}4 \in R_{>1}$.

Moreover, x is a cluster root, for all $-1 <_\nu x <_\nu 2$.

Example 4.27. Consider the uniform layered semiring† $R = (\mathbb{N}^*, \mathbb{T}^*)$ and its polynomial

$$F = X^3 \oplus 3X^2 \oplus 4X \in R[X],$$

where the layer of all coefficients is 1, so we have not written it visible. Now, all ghost elements $a \in R_{>1}$ satisfy $F(a) \in R_{>1}$. However, those ghost elements that are ν-equivalent to 1 or 3 are corner roots, and thus, these points have two dominating monomials. Therefore

$$\mathscr{Z}_{\text{clus}}(F) = \{a \in R_{>1} \mid a \not\cong_\nu 1 \text{ and } a \not\cong_\nu 3\}.$$

Like corner roots, also cluster roots satisfy the following property (in the case of several layers).

Lemma 4.28. *Let L be a sorting semiring†, R an L-layered semiring†, and $F \in R[X_1, \dots, X_n]$. If $a \in R^n$ is a cluster root of F, then*

$$F(a) \in R_{>1}.$$

Proof. Write $F = \sum_{i=1}^{r} F_i$. As shown in Example 4.25, there is no cluster roots, if $L = \{1\}$. Therefore we can assume that $|L| > 1$.

If $a \in \mathcal{Z}_{\text{clus}}(F)$, then $F_i(a) \in R_{>1}$, for $i \in \{1, \dots, r\}$, and $F(a) \not\cong_\nu F_j(a)$ for all $j \neq i$. Based on the remark after Definition 4.24, the latter condition implies $F(a) = F_i(a)$. By taking into account the former condition, we obtain $F(a) \in R_{>1}$. □

Until now we have seen two kinds of roots: corner roots and cluster roots. As a combination of them, we give the following definition [18, pp. 28–29] (Definition 7.22).

Definition 4.29. Let L be a sorting semiring†, R an L-layered semiring†, $S \subset R^n$ a subset, and $F \in R[X_1, \dots, X_n]$. The *total locus* with respect to S is

$$\mathcal{Z}(F; S) := \mathcal{Z}_{\text{corn}}(F; S) \cup \mathcal{Z}_{\text{clus}}(F; S).$$

When $S = R^n$, we write $\mathcal{Z}(F)$ for $\mathcal{Z}(F; R^n)$.

The elements of total locus are called *roots*.

Remark. The term "total locus" is taken from [21]. The earlier name of the same concept is "combined locus", existing in all the publication of the same authors earlier than [21] (from the year 2014). The earlier denotation for $\mathcal{Z}(F; S)$ was $\mathcal{Z}_{\text{comb}}(F; S)$.

Total locus is a disjoint union of corner locus and cluster locus, since the former subset consists of points, where several monomials dominate, and the latter one includes only such points, where a single monomial dominates.

Example 4.30. Let L be a sorting semiring†, R an L-layered semiring† $S \subset R^n$ a subset, and $F \in R[X_1, \dots, X_n]$. If $L = \{1\}$, then $\mathcal{Z}_{\text{clus}}(F; S) = \emptyset$ (as already shown in Example 4.25), and thus, $\mathcal{Z}(F; S) = \mathcal{Z}_{\text{corn}}(F; S)$.

The following lemma gives an alternative description for a root in the case of several layers.

Lemma 4.31. *Let L be a sorting semiring† such that $|L| > 1$, R an L-layered semiring†, $S \subset R^n$ a subset, and $F \in R[X_1, \dots, X_n]$. Then*

$$a \in \mathcal{Z}(F; S) \quad \Longleftrightarrow \quad F(a) \in R_{>1}.$$

Proof. "$\Rightarrow$" If $a \in \mathcal{Z}(F; S)$, then $a \in \mathcal{Z}_{\text{corn}}(F; S)$ or $a \in \mathcal{Z}_{\text{clus}}(F; S)$. Since $|L| > 1$, the claim follows from Lemmata 4.23 and 4.28.

"$\Leftarrow$" Suppose that $a \in S$ such that $F(a) \in R_{>1}$. We always have the situation that either one or several monomials dominate at each point. Assume first that a single monomial dominates at a, and let F_i denote this monomial. Therefore $F_i(a) = F(a)$, and thus, $F_i(a) \in R_{>1}$. Since F_i is the only monomial dominating at a, the other monomials do not dominate, and thus, $F(a) >_\nu F_j(a)$, for all $j \neq i$. This implies $F(a) \not\cong_\nu F_j(a)$, for all $j \neq i$. Hence $a \in \mathcal{Z}_{\text{clus}}(F; S) \subset \mathcal{Z}(F; S)$.

Assume next that there are several (at least two) monomials dominating at a. This means that $a \in \mathcal{Z}_{\text{corn}}(F; S) \subset \mathcal{Z}(F; S)$. □

Remark. In the case of several layers, roots are exactly the points, where a polynomial reaches a ghost value (recall the remark after Definition 3.23). On the other hand, roots are exactly the points, where the values of a polynomial comprise an ideal (recall Proposition 3.29). This is how ghost elements can play the role of the missing zero element in a semiring†.

Ghost values are most often roots, for example if the polynomial lacks the constant term (as shown in Example 4.27) or if the constant term does not dominate at the root. This means that ghost values are not so interesting as roots, and thus, we will mainly pay attention to tangible roots.

4.4 Alternative root definitions

Recall Definition 4.21 for a corner root, as well as its interpretation, explained in the remark after the definition. There exist alternative definitions for the same concept, but the definitions are not equivalent. One of such definitions is the following one that is taken from [15] (Definition 5.16).

Definition 4.32 (Not in actual use in this work)**.** Let R be a layered semiring†, and $F \in R[X_1, \dots, X_n]$, written as $F = \sum_{i=1}^{r} F_i$. An element $a \in R^n$ is a *corner root* of F, if $F(a) \neq F_i(a)$ for all $i \in \{1, \dots, r\}$.

The following lemma shows that Definitions 4.21 and 4.32 are equivalent, if the sorting semiring† in question has no infinite elements.

Lemma 4.33. *Let L be a sorting semiring†, R an L-layered semiring†, $a \in R^n$, and $F \in R[X_1, \dots, X_n]$, written as $F = \sum_{i=1}^{r} F_i$. If L comprises no infinite elements, then the following conditions are equivalent:*

(i) $F(a) \neq F_i(a)$ *for all* $i \in \{1, \dots, r\}$*,*

(ii) $F(a) \cong_\nu F_i(a) \cong_\nu F_j(a)$*, for* $i, j \in \{1, \dots, r\}$ *such that* $i \neq j$*.*

Proof. The claim follows from Lemma 4.14, since at least one monomial dominates at each point. □

Remark. Consider the case, where L has an infinite element. For example, if $L = \{1\}$, then 1 is infinite. If now F has two monomials dominating at a, say F_i and F_j, then $F(a) = F_i(a) = F_j(a)$, which reveals that the above conditions are not equivalent. The same thing is possible, if L has several elements with an infinite one among them.

Another alternative definition for a corner root can be found in [21, p. 8] (Definition 4.1). This definition is based on tangible lift, the definition of which is given in [21, p. 5], as follows.

Definition 4.34. Let R be a layered semiring†, and $a \in R$. Choose an element $\hat{a} \in R_1$ such that $\hat{a} \cong_\nu a$. Then the map

$$R \to R_1, \quad a \mapsto \hat{a}$$

is called a *tangible lift*.

If F is a polynomial over R, then $\hat{F}$ is a polynomial, derived from F by applying tangible lift to the coefficient of each monomial in F. More precisely, if $F = \sum_{i=1}^{r} F_i$, written as a sum of monomials, then

$$\hat{F} = \widehat{\sum_{i=1}^{r} F_i} = \sum_{i=1}^{r} \hat{F}_i,$$

when $\hat{F}_i$ is a monomial in $\hat{F}$, for all $i \in \{1, \dots, r\}$.

Remark. In [21], R is assumed to be a semidomain† and to have at least two layers, but such requirements are not necessary here.

For tangible elements, tangible lift can be assumed to be the identity map. Otherwise tangible lift gives the preimage of a sort transition map, $R_1 \to R_{>1}$, which is supposed to be surjective (as mentioned in the remark after Definition 3.23). Since this map is not necessarily injective, there can be several choices for $\hat{a}$. In the case of a uniform layered semiring†, the preimage is unique.

Tangible lift can be used in defining a corner root in [21, p. 8] (Definition 4.1) as follows.

Definition 4.35 (Not in actual use in this work)**.** Let R be a layered semiring†, and $F \in R[X_1, \dots, X_n]$. An element $a \in R^n$ is a *corner root* of F, if $\hat{F}(\hat{a}) \in R_{>1}$.

The following lemma shows that Definitions 4.21 and 4.35 are equivalent, if the sorting semiring† has at least two elements. Such an assumption is necessary, since otherwise condition (i) in the lemma could never hold true.

Lemma 4.36. *Let R be a layered semiring† such that $|L| > 1$, $a \in R^n$, and $F \in R[X_1, \dots, X_n]$, written as $F = \sum_{i=1}^{r} F_i$. The following conditions are equivalent:*

(i) $\hat{F}(\hat{a}) \in R_{>1}$,

(ii) $F(a) \cong_\nu F_i(a) \cong_\nu F_j(a)$*, for* $i, j \in \{1, \ldots, r\}$ *such that* $i \neq j$.

Proof. "(i) $\Rightarrow$ (ii)" Since $\hat{a}$ as well as all the coefficients of $\hat{F}$ are tangible, the condition in (i) can hold true only when $\hat{F}$ has at least two monomials dominating at $\hat{a}$, i.e.

$$\hat{F}(\hat{a}) \cong_\nu \hat{F}_i(\hat{a}) \cong_\nu \hat{F}_j(\hat{a}),$$

for some distinct monomials $\hat{F}_i$ and $\hat{F}_j$ in $\hat{F}$. Based on the definition of tangible lift, $\hat{F}_i(\hat{a}) \cong_\nu F_i(a)$, for all $i \in \{1, \ldots, r\}$, and thus, the above ν-equivalences imply

$$F(a) \cong_\nu F_i(a) \cong_\nu F_j(a),$$

for some distinct monomials F_i and F_j in F. This implies (ii).

"(ii) $\Rightarrow$ (i)" Based on the definition of tangible lift, $\hat{F}_i(\hat{a}) \cong_\nu F_i(a)$, for all $i \in \{1, \ldots, r\}$ Therefore the ν-equivalences in (ii) imply

$$\hat{F}(\hat{a}) \cong_\nu \hat{F}_i(\hat{a}) \cong_\nu \hat{F}_j(\hat{a}),$$

for $i, j \in \{1, \ldots, r\}$ such that $i \neq j$. Therefore $\hat{F}$ has at least two monomials dominating at $\hat{a}$, which implies (i). □

Remark. In the latter direction of the above proof, we have the assumption that $F_i(a) \cong_\nu F_j(a)$, and we apply tangible lift for both sides of the ν-equivalence. This means that we apply tangible lift twice to a. Since R is not required to be uniform, we can end up to different tangible values at different times.

More precisely, suppose that $a_1, a_2 \in R_1$ such that $a \cong_\nu a_1$ and $a \cong_\nu a_2$ but $a_1 \neq a_2$. Applying tangible lift to $F_i(a) \cong_\nu F_j(a)$ may yield $\hat{F}_i(a_1) \cong_\nu \hat{F}_j(a_2)$. From this ν-equivalence, we cannot conclude that $\hat{F}$ has at least two monomials dominating at the same point.

However, tangible lift yields ν-equivalences $\hat{F}_i(\hat{a}) \cong_\nu \hat{F}_j(\hat{a})$ for all $\hat{a} \in R_1$ such that $\hat{a} \cong_\nu a$. Among them, we can find the one, which has the same $\hat{a}$ on the both sides. Therefore we can conclude that $\hat{F}$ has at least two monomials dominating at $\hat{a}$. (Actually, $\hat{F}$ has at least two monomials dominating at all those points $\hat{a}$ that satisfy $\hat{a} \cong_\nu a$.)

After all, the only deficience in Definition 4.35 is that it requires the sorting semiring† to have at least two elements. Such a requirement is not needed in Definition 4.21, and thus, we prefer it as a definition for a corner root. Although we reject tangible lift in the definition of a corner root, it will later turn out to be practical in other contexts.

Chapter 5

Towards congruence varieties

5.1 Affine layered algebraic sets

This chapter introduces (somewhat miscellaneous) basic elements for congruence varieties, which is the main subject of the next chapter. One of such basic element is an algebraic set. In usual algebraic geometry, algebraic sets are determined by the common roots of a certain set of polynomials. (Such a set of polynomials typically comprises an ideal). In tropical world, we have a corner root set as given in Definition 4.21 and a total root set as given in Definition 4.29. Based on them, we define algebraic sets as follows [18, p. 29], [21, p. 9].

Definition 5.1. Let R be a layered semiring†, $S \subset R^n$ a subset, and $I \subset R[X_1, \ldots, X_n]$ a subset. The *(affine) corner algebraic set* and the *(affine) (total) algebraic set* of I with respect to S are, respectively,

$$\mathcal{Z}_{\mathrm{corn}}(I; S) := \bigcap_{F \in I} \mathcal{Z}_{\mathrm{corn}}(F; S) \quad \text{and} \quad \mathcal{Z}(I; S) := \bigcap_{F \in I} \mathcal{Z}(F; S).$$

If $S = R^n$, we write $\mathcal{Z}_{\mathrm{corn}}(I)$ for $\mathcal{Z}_{\mathrm{corn}}(I; R^n)$, and $\mathcal{Z}(I)$ for $\mathcal{Z}(I; R^n)$.

Remark. The subset I above is not necessarily an ideal. It is allowed to be infinite (as well as finite).

In the case of a single layer, $\mathcal{Z}(F; S) = \mathcal{Z}_{\mathrm{corn}}(F; S)$, as shown in Example 4.30. In this case also $\mathcal{Z}(I; S) = \mathcal{Z}_{\mathrm{corn}}(I; S)$.

Example 5.2. Let R be a layered semiring†. Consistently with usual algebraic geometry, the empty set is an algebraic set, since $\mathcal{Z}(R[X_1, \ldots, X_n]) = \emptyset$.

Moreover, $\mathcal{Z}(\{a\}) = \emptyset$, for all $a \in R_1$, i.e. for all tangible-valued constant polynomials. On the other hand, $\mathcal{Z}(\{a\}) = R^n$, for all $a \in R_{>1}$, i.e. for all ghost-valued constant polynomials.

Example 5.3. The set consisting of a single tangible point is a corner algebraic set. Namely, if R is a layered semiring† and $a = (a_1, \ldots, a_n) \in R_1^n$, then

$$\mathcal{Z}_{\text{corn}}(\{X_1 + a_1, \ldots, X_n + a_n\}; R_1^n) = \{a\}.$$

On the other hand, $\mathcal{Z}_{\text{corn}}(\{X_1 + a_1, \ldots, X_n + a_n\})$ consists of a and all those elements of R that are ν-equivalent to a.

Example 5.4. In Chapter 2, we have seen examples of corner algebraic sets, e.g. in Figure 2.3. More formally, if R is a layered semiring† such that $R_1 = \mathbb{T}$, and

$$F = aX + bY + c \in R_1[X, Y],$$

then the tangible corner algebraic set, $\mathcal{Z}_{\text{corn}}(\{F\}; R_1^2)$ determines a tropical line.

On the other hand, if $F \in R_1[X, Y]$ is a binomial, then the corner algebraic set, $\mathcal{Z}_{\text{corn}}(\{F\}; R_1^2)$ determines a usual line (in $\mathbb{R}^2$).

Example 5.5. Let R be a layered semiring† such that $R_1 = \mathbb{T}$, and

$$F = X + Y + a \quad \text{and} \quad G = X + Y + b$$

elements of $R_1[X, Y]$. If $a <_\nu b$, then

$$\mathcal{Z}_{\text{corn}}(\{F, G\}; R_1^2) = \{(c, c) \in R_1^2 \mid c \geq_\nu b\}.$$

In other words, the above corner algebraic set determines a ray, the upper part of the line $y = x$, with (b, b) as the endpoint of the ray.

Remark. Similarly, a line segment is a corner algebraic set [21, p. 8] (Example 4.4). In the case of line segments and rays, the endpoints are always common corner roots. Therefore, a corner algebraic set can determine only a closed line segment or ray.

Two tropical polynomials can have a common corner root such that the polynomials do not reach the same value at the point of the corner root, as discussed in the remark after Example 2.23. This will be shown also in the following example.

Example 5.6. Let

$$F = X \oplus 0 \quad \text{and} \quad G = 1X \oplus 1$$

be elements of $\mathbb{T}[X]$. Now 0 is the common corner root of both F and G, but $F(0) = 0$ and $G(0) = 1$. However, if the above polynomials are elements of $\mathcal{R}(\mathbb{N}^*, \mathbb{T}^*)[X]$, then both $F(0)$ and $G(0)$ are ghosts.

Example 5.7. Let L be a sorting semiring† and R a uniform L-layered semiring†. Consider the polynomial semiring† $R[X,Y,Z]$. If $L = \{1,\infty\}$, it holds true that

$$(X \oplus Y)(X \oplus Z)(Y \oplus Z) = (X \oplus Y \oplus Z)(XY \oplus XZ \oplus YZ).$$

Namely, the only difference between these two polynomials is that the term XYZ occurs twice on the left and three times on the right, and such a difference has no effect when $L = \{1,\infty\}$.

Denote the left side as F and the right side as G. Since $F = G$ (in the case of $L = \{1,\infty\}$), then clearly

$$\mathcal{Z}(F) = \mathcal{Z}(G).$$

The equation between the algebraic sets holds true even if $L \neq \{1,\infty\}$. Namely, the monomial XYZ dominates F and G only at those points that are ν-equivalent to (a,a,a), for $a \in R_1$, and

$$\begin{aligned} F(a,a,a) &= {}^{[2]}a \odot {}^{[2]}a \odot {}^{[2]}a = {}^{[8]}a^3, \\ G(a,a,a) &= {}^{[3]}a \odot {}^{[3]}a^2 = {}^{[9]}a^3, \end{aligned}$$

when assuming, for example, that $L = \mathbb{N}^*$. Therefore the elements that are ν-equivalent to (a,a,a) are included in $\mathcal{Z}(F)$ as well as in $\mathcal{Z}(G)$.

5.2 Tropical regions

The next basic element for congruence varieties is a tropical region. As mentioned in Examples 2.21 and 2.22. the graph of a tropical polynomial function divides a plane (or more generally, a space) into regions determined by each essential monomial in the polynomial defining the polynomial function. We will next introduce the concept of a region more precisely, inspired by [19, p. 26] and [21, p. 9]. Since regions can be considered from the geometric point of view, we assume the algebraic structure to be a 1-semifield†.

Definition 5.8. Let K be a 1-semifield†, and $F \in K[X_1,\ldots,X_n]$. Write F as a sum of monomials, i.e. $F = \sum_{i=1}^{r} F_i$. Then for each $i \in \{1,\ldots,r\}$, the set

$$D_{F,i} := \{a \in K^n \mid F(a) \not\cong_\nu F_j(a) \text{ for all } j \neq i\},$$

where $j \in \{1,\ldots,r\}$, is called an *(open) tropical region.* A *closed tropical region* is defined as

$$\overline{D}_{F,i} := \{a \in K^n \mid F(a) \cong_\nu F_i(a)\}.$$

Cutting the above sets by K_1^n gives sets called an *(open) tangible tropical region* and a *closed tangible tropical region*, respectively.

Remark. An open tropical region can be the empty set, since inessential monomials are allowed in the above definition. Otherwise an open tropical region consists of the points, where a single monomial, F_i dominates. Namely, the condition $F(a) \not\cong_\nu F_j(a)$ for all $j \neq i$ is equivalent to $F(a) >_\nu F_j(a)$ for all $j \neq i$. This means that none of the monomials F_j dominates F at a, but since there always has to be at least one dominating monomial, the dominating monomial must be F_i.

Further, if a single monomial, F_i dominates F at a, then $F(a) = F_i(a)$, which was discussed in the remark after Lemma 4.14. However, the opposite implication does not hold, if the layer of a or the layer of the coefficient of F_i is infinite. This was discussed in the remark after Definition 4.24.

Definition 5.8 uses the same denotation as has been done in [21, p. 9] (Definition 4.8) for a *component*, defined as

$$D_{F,i} := \{a \in K^n \mid \hat{F}(\hat{a}) = \hat{F}_i(\hat{a})\},$$

by applying tangible lift (introduced in Definition 4.34). However, the above kind of component is not suitable for our purposes, since it does not recognize correctly the situations, where a single monomial dominates. The above definition works well, if the layered semiring† is uniform and if the sorting semiring† consists of more than one element. This is proved in the following lemma.

Lemma 5.9. *Let L be a sorting semiring† such that $|L| > 1$, R a uniform L-layered semiring†, $a \in R^n$, and $F \in R[X_1, \dots, X_n]$, written as $\sum_{i=1}^{r} F_i$. The following conditions are equivalent:*

(i) $\hat{F}(\hat{a}) = \hat{F}_i(\hat{a})$.

(ii) $F(a) \not\cong_\nu F_j(a)$ *for all* $j \in \{1, \dots, r\}$ *such that* $j \neq i$.

Proof. "(i) $\Rightarrow$ (ii)" Since $|L| > 1$ and since the equation in (i) concerns tangible values, we can conclude that $\hat{F}_i$ is the only monomial in $\hat{F}$ dominating at $\hat{a}$. In a uniform layered semiring†, this means that the actual value of $\hat{F}_i(\hat{a})$ is strictly greater than those of $\hat{F}_j(\hat{a})$, for all $j \in \{1, \dots, r\}$ such that $j \neq i$. Since the sort transition maps, as well as tangible lift, preserve the actual values in the uniform case, we can conclude that also F_i is the only monomial in F dominating at a. This implies (ii).

"(ii) $\Rightarrow$ (i)" As discussed in the remark after Definition 5.8, (ii) implies $F(a) = F_i(a)$, and thus,

$$\hat{F}(\hat{a}) \cong_\nu F(a) = F_i(a) \cong_\nu \hat{F}_i(\hat{a}).$$

This gives the ν-equivalence $\hat{F}(\hat{a}) \cong_\nu \hat{F}_i(\hat{a})$, but since R is uniform and both sides of the ν-equivalence are tangible, i.e. share the same layer, we actually have equation $\hat{F}(\hat{a}) = \hat{F}_i(\hat{a})$. $\square$

Remark. The assumption that the semiring† is uniform is necessary in the first direction of the above proof, but not in latter one. Namely, the latter direction could be proved alternatively as follows:

Now $\hat{F}_i(\hat{a}) \cong_\nu F_i(a)$ for all $i \in \{1, \dots, r\}$, and thus, (ii) implies that $\hat{F}(\hat{a}) \not\cong_\nu \hat{F}_j(\hat{a})$ for all $j \in \{1, \dots, r\}$ such that $j \neq i$. Therefore $\hat{F}_i$ is the only possible monomial in $\hat{F}$ dominating at $\hat{a}$. Based on Definition 4.34, $\hat{F}_i$ is indeed a monomial in $\hat{F}$, and thus, $\hat{F}(\hat{a}) = \hat{F}_i(\hat{a})$. Q.E.D.

The equation $\hat{F}(\hat{a}) = \hat{F}_i(\hat{a})$ may hold also in the case of a corner root, if $L = \{1\}$ or if R is not uniform. Namely, if $L = \{1\}$, then tangible lift is an identity map, and thus, $\hat{F}(\hat{a}) = \hat{F}_i(\hat{a})$ means the same as $F(a) = F_i(a)$. But it is still possible that a is a corner root, and thus, (i) does not imply (ii). In the case where R is not uniform, suppose that $F = F_1 + F_2$ is a polynomial over R such that $F_1(a) \cong_\nu F_2(a)$. Therefore $a \in R^n$ is a corner root of F. By applying tangible lift, we have

$$\hat{F}_1(\hat{a}) \cong_\nu F_1(a) \cong_\nu F_2(a) \cong_\nu \hat{F}_2(\hat{a}),$$

when it is possible that $\hat{F}_1(\hat{a}) \neq \hat{F}_2(\hat{a})$. Depending on the mutual order between these elements, it holds either $\hat{F}(\hat{a}) = \hat{F}_1(\hat{a})$ or $\hat{F}(\hat{a}) = \hat{F}_2(\hat{a})$.

Now we can move to examples on tropical regions.

Example 5.10. Let K be a 1-semifield†. The whole space K^n is a tropical region. Namely, this occurs, when $F \in K[X_1, \dots, X_n]$ is a monomial. If F is a binomial, both its monomials define a half-space, which is therefore a tropical region.

If $n = 1$, an open tangible interval is a tangible tropical region, while a closed tangible interval is a closed tangible tropical region. If $n = 2$, a tangible polygon is a closed tangible tropical region. To see this more precisely, consider the following polynomial and its monomial

$$F = X^2Y^2 \oplus X^2 \oplus Y^2 \oplus 1XY \oplus 0 \quad \text{and} \quad F_1 = 1XY,$$

the elements of $K[X,Y]$. Then $\overline{D}_{F,1}$ is a square, depicted in [21, p. 13] (Figure 3 a). Most often tropical regions of case $n = 2$ are polygons unbounded in one side, as can be seen Figures 2.3 and 2.4.

If monomials in a polynomial have ghost coefficients, the tropical regions determined by these monomials correspond to the cluster roots of the polynomial. This relationship between cluster roots and tropical regions is described in the following lemma. It utilizes the shorter notation for the indeterminants, as introduced in Definition 4.8.

Lemma 5.11. *Let K be a 1-semifield†, and $F \in K[X_1, \dots, X_n]$, written as $F = \sum_{i=1}^r F_i$. Then*

$$\mathcal{Z}_{\text{clus}}(F) = \bigcup_{F_i \in K_{>1}[\boldsymbol{X}]} D_{F,i},$$

when taking only tangible roots into account and $D_{F,i}$ refers to the tangible tropical region

$$D_{F,i} = \{a \in K_1^n \mid F(a) \not\cong_\nu F_j(a) \text{ for all } j \neq i\}.$$

Proof. Suppose that $a \in K_1^n$ is a cluster root of F. Then there exists $i \in \{1, \dots, r\}$ such that F_i is a single monomial dominating at a and $F_i(a) \in K_{>1}$. Since a is tangible, the coefficient of F_i must be ghost-valued, and thus, the latter condition can be written as $F_i \in K_{>1}[X_1, \dots, X_n]$. Hence, $a \in D_{F,i}$ for some $i \in \{1, \dots, r\}$ such that $F_i \in K_{>1}[X_1, \dots, X_n]$.

The other direction is even more straightforward. $\square$

Remark. Each set in the above lemma can be the empty set.

As expected, an open tropical region is a subset of a closed tropical region, if these sets are determined by the same monomial of a polynomial. This will be proved in the following lemma.

Lemma 5.12. *Let K be a 1-semifield†, and $F \in K[X_1, \dots, X_n]$, written as $F = \sum_{i=1}^r F_i$. Then*

$$D_{F,i} \subset \overline{D}_{F,i},$$

for all $i \in \{1, \dots, r\}$.

Proof. If $D_{F,i} = \emptyset$, the claim is true. If this is not the case, suppose that $a \in D_{F,i}$. Then $F(a) \not\cong_\nu F_j(a)$ for all $j \neq i$. Therefore $F(a) = F_i(a)$, and thus, $F(a) \cong_\nu F_i(a)$. This means that $a \in \overline{D}_{F,i}$ $\square$

The above subset is proper, except for the case of monomials, which have no corner roots. This is clarified in the following lemma.

Lemma 5.13. *Let K be a 1-semifield†, and $F \in K[X_1, \dots, X_n]$, written as $F = \sum_{i=1}^r F_i$. Then*

$$\bigcup_{i=1}^r (\overline{D}_{F,i} \backslash D_{F,i}) = \mathcal{Z}_{\text{corn}}(F),$$

for all $i \in \{1, \dots, r\}$.

Proof. If $a \in K^n$, then

$$\begin{aligned}
& a \in \bigcup_{i=1}^r (\overline{D}_{F,i} \backslash D_{F,i}) \\
\iff & a \in \overline{D}_{F,i} \backslash D_{F,i}, \text{ for } i \in \{1, \dots, r\} \\
\iff & a \in \overline{D}_{F,i} \quad \text{and} \quad a \notin D_{F,i}, \text{ for } i \in \{1, \dots, r\} \\
\iff & F(a) \cong_\nu F_i(a), \text{ for } i \in \{1, \dots, r\} \quad \text{and} \quad F(a) \cong_\nu F_j(a) \text{ for } j \neq i \\
\iff & F(a) \cong_\nu F_i(a) \cong_\nu F_j(a), \text{ for } i, j \in \{1, \dots, r\} \text{ such that } j \neq i \\
\iff & a \in \mathcal{Z}_{\text{corn}}(F).
\end{aligned}$$

$\square$

Corner roots can be expressed in another way, too, as follows.

Lemma 5.14. *Let K be a 1-semifield†, and $F \in K[X_1, \dots, X_n]$, written as $F = \sum_{i=1}^{r} F_i$. Then*
$$\mathcal{Z}_{\mathrm{corn}}(F) = \bigcup_{i \neq j} (\overline{D}_{F,i} \cap \overline{D}_{F,j}).$$

Proof. If $a \in K^n$, then

$$\begin{aligned} & a \in \bigcup_{i \neq j} (\overline{D}_{F,i} \cap \overline{D}_{F,j}) \\ \iff & a \in \overline{D}_{F,i} \cap \overline{D}_{F,j}, \text{ for distinct } i, j \in \{1, \dots, r\} \\ \iff & a \in \overline{D}_{F,i} \text{ and } a \in \overline{D}_{F,j}, \text{ for distinct } i, j \in \{1, \dots, r\} \\ \iff & F(a) \cong_\nu F_i(a) \text{ and } F(a) \cong_\nu F_j(a), \text{ for distinct } i, j \in \{1, \dots, r\} \\ \iff & a \in \mathcal{Z}_{\mathrm{corn}}(F). \end{aligned}$$

□

Total root set can be expressed as follows, when again using the shorter notation for indeterminants, as introduced in Definition 4.8.

Lemma 5.15. *Let K be a 1-semifield†, and $F \in K[X_1, \dots, X_n]$, written as $F = \sum_{i=1}^{r} F_i$. Then*

$$\mathcal{Z}(F) = \bigcup_{i \neq j \ \text{or} \ F_i \in K_{>1}[\boldsymbol{X}]} (\overline{D}_{F,i} \cap \overline{D}_{F,j}).$$

when taking only tangible roots into account and $\overline{D}_{F,i}$ refers to the closed tangible tropical region for all $i \in \{1, \dots, r\}$.

Proof. As the closed tropical region in the claim, also $D_{F,i}$ below refers to the tangible tropical region for all $i \in \{1, \dots, r\}$.

As given in Definition 4.29, the total root set of a polynomial is a (disjoint) union of its corner and cluster root sets. Therefore by applying Lemmata 5.13 and 5.11, we obtain

$$\mathcal{Z}(F) = \mathcal{Z}_{\mathrm{corn}}(F) \cup \mathcal{Z}_{\mathrm{clus}}(F) = \bigcup_{i=1}^{r} (\overline{D}_{F,i} \backslash D_{F,i}) \cup \bigcup_{F_i \in K_{>1}[\boldsymbol{X}]} D_{F,i}.$$

We have above a union of two unions, the first of which is taken over all monomials in F, and the other one only over a subset of them. We can join the unions to a single union by using the notation:

$$\tilde{D}_{F,i} = \begin{cases} D_{F,i}, & \text{if } F_i \in K_{>1}[\boldsymbol{X}], \\ \emptyset, & \text{otherwise.} \end{cases}$$

Therefore we can continue as follows

$$\begin{aligned}
\mathcal{Z}(F) &= \bigcup_{i=1}^{r}(\overline{D}_{F,i}\backslash D_{F,i}) \cup \bigcup_{F_i\in K_{>1}[\boldsymbol{X}]} D_{F,i} \\
&= \bigcup_{i=1}^{r}(\overline{D}_{F,i}\backslash D_{F,i}) \cup \bigcup_{i=1}^{r} \tilde{D}_{F,i} \\
&= \bigcup_{i=1}^{r}(\overline{D}_{F,i}\backslash D_{F,i} \cup \tilde{D}_{F,i}) \\
&= \bigcup_{F_i\notin K_{>1}[\boldsymbol{X}]}(\overline{D}_{F,i}\backslash D_{F,i} \cup \emptyset) \cup \bigcup_{F_i\in K_{>1}[\boldsymbol{X}]}(\overline{D}_{F,i}\backslash D_{F,i} \cup D_{F,i}) \\
&= \bigcup_{F_i\notin K_{>1}[\boldsymbol{X}]}(\overline{D}_{F,i}\backslash D_{F,i}) \cup \bigcup_{F_i\in K_{>1}[\boldsymbol{X}]} \overline{D}_{F,i}.
\end{aligned}$$

Since the closed sets $\overline{D}_{F,i}$ are included in every case, we can take the first union over all monomials. Therefore we can continue as follows

$$\begin{aligned}
\mathcal{Z}(F) &= \bigcup_{F_i\notin K_{>1}[\boldsymbol{X}]}(\overline{D}_{F,i}\backslash D_{F,i}) \cup \bigcup_{F_i\in K_{>1}[\boldsymbol{X}]} \overline{D}_{F,i} \\
&= \bigcup_{i=1}^{r}(\overline{D}_{F,i}\backslash D_{F,i}) \cup \bigcup_{F_i\in K_{>1}[\boldsymbol{X}]} \overline{D}_{F,i} \\
&= \mathcal{Z}_{\mathrm{corn}}(F) \cup \bigcup_{F_i\in K_{>1}[\boldsymbol{X}]} \overline{D}_{F,i} \\
&= \bigcup_{i\neq j}(\overline{D}_{F,i} \cap \overline{D}_{F,j}) \cup \bigcup_{F_i\in K_{>1}[\boldsymbol{X}]} \overline{D}_{F,i} \\
&= \bigcup_{i\neq j \text{ or } F_i\in K_{>1}[\boldsymbol{X}]}(\overline{D}_{F,i} \cap \overline{D}_{F,j}).
\end{aligned}$$

Note that we have above applied both Lemma 5.13 and Lemma 5.14, which give different representations for the set of corner roots. □

The next lemma shows that a closed tropical region can be expressed as a total root set of certain polynomials.

Lemma 5.16. *Let K be a 1-semifield† with at least two layers. Suppose that $F \in K[X_1, \ldots, X_n]$, and F_i is a monomial in F. The closed tangible tropical region*

$$\overline{D}_{F,i} := \{a \in K_1^n \mid F(a) \cong_\nu F_i(a)\}$$

is a total algebraic set, when taking only tangible roots into account.

Proof. Consider a total algebraic set

$$\mathcal{Z}(I) = \bigcap_{H\in I} \mathcal{Z}(H),$$

for some subset $I \subset K[X_1, \ldots, X_n]$.

We will next show how to construct I in a way that proves the claim. If F_i is the only monomial in F, then the claim follows by setting $I := \{F_i + F_i\}$. Namely, now

$$\overline{D}_{F,i} = K_1^n = \mathcal{Z}(F_i + F_i).$$

Otherwise write $F = \sum_{j=1}^r F_j$. By applying tangible lift to the coefficients of F, we define

$$I := \bigcup_{\substack{j=1 \\ j \neq i}}^{r} \{F_i + F_i + \hat{F}_j\},$$

and, show again that

$$\overline{D}_{F,i} = \mathcal{Z}(I).$$

Suppose first that $a \in \overline{D}_{F,i}$, which means that $F(a) \cong_\nu F_i(a)$. In other words, F_i dominates F at a, and thus, F_i dominates also each $F_i + F_i + \hat{F}_j$ at a. Therefore, $(F_i + F_i + \hat{F}_j)(a) \in K_{>1}$, for all $j \in \{1, \ldots, i-1, i+1, \ldots, r\}$. This means that a is a root of all $F_i + F_i + \hat{F}_j$. Hence $a \in \mathcal{Z}(I)$.

Suppose next that $a \in \mathcal{Z}(I)$ for some tangible $a \in K_1^n$. This means that a is a root of each $F_i + F_i + \hat{F}_j$. Note that these three summands actually form a binomial consisting of the monomials $F_i + F_i$ and $\hat{F}_j$. If a is a corner root of some binomial $F_i + F_i + \hat{F}_j$, then both the monomials, $F_i + F_i$ and $\hat{F}_j$ dominate at a. If a is a cluster root of some binomial $F_i + F_i + \hat{F}_j$, then $F_i + F_i$ must be the dominating monomial. This is due to the tangible coefficients of all $\hat{F}_j$ and the assumption that a is tangible. In both cases, $F_i + F_i$ is a monomial dominating the binomial at a (either alone or with some $\hat{F}_j$). More precisely,

$$(F_i + F_i)(a) \geq_\nu \hat{F}_j(a),$$

for all $j \in \{1, \ldots, r\}$ (and $j \neq i$). Furthermore

$$F_i(a) \cong_\nu (F_i + F_i)(a) \geq_\nu \hat{F}_j(a) \cong_\nu F_j(a),$$

for all $j \in \{1, \ldots, r\}$ (and $j \neq i$). Therefore $F_i(a) \cong_\nu F_i(a) + F_j(a)$, for all $j \in \{1, \ldots, r\}$ and $j \neq i$, and thus, $F_i(a) \cong_\nu F(a)$. Hence, $a \in \overline{D}_{F,i}$. □

5.3 Congruences

As a third basic element for congruence varieties, we will introduce congruences. Although the concept of an ideal is important with rings in commutative algebra, and although we have already seen ideals with semirings, this section (as well as the rest of this chapter) shows how congruences fit better than ideals in with semirings. We will consider congruences in general, not in the layered context, and thus, we reject semirings† and turn to semirings. We will introduce even such cases, where the zero element turns out to be necessary. However, these cases are not totally necessary themselves later in the context of congruence varieties.

Although the missing zero in a semiring† can sometimes be replaced by ghost elements (as discussed in the remark after Lemma 4.31), the problem is that ghosts do not act as a neutral element of addition operation. Therefore ghost elements cannot properly take the place of the zero element.

The problem with ideals in a semiring arises as follows. In ring theory, quotient structures can be determined by ideals. This is not possible with semirings, since the equality between cosets is defined as

$$a + I = b + I \quad \Longleftrightarrow \quad a - b \in I,$$

where a and b are the elements of a ring, and I is its ideal. The problem with a semiring is that there is not necessarily such an element as $-b$. Thus, we need to use congruence relations for defining quotient semirings.

Definition 5.17. Let $(R, +, \cdot)$ be a semiring, and $\equiv$ an equivalence relation on R. Let $a, b, c, d \in R$ such that $a \equiv b$ and $c \equiv d$. If

$$a + c \equiv b + d \quad \text{and} \quad a \cdot c \equiv b \cdot d,$$

then $\equiv$ is said to be a *congruence (relation)* on R. Whenever $a \equiv b$, it is said that a and b are *congruent*, or a is *congruent* with b.

Remark. If a and b are congruent in respect to a congruence relation Ω, then $(a, b) \in \Omega$, and we can consider a congruence relation as a set, a subset of $R \times R$. A congruence Ω is a *proper congruence (on R)*, if $\Omega \subsetneq R \times R$.

Example 5.18. Let R be a ring, $a, b \in R$, and $I \subset R$ an ideal. Consider the aforementioned relation

$$a + I = b + I \quad \Longleftrightarrow \quad a - b \in I,$$

and define the set

$$\Omega := \{(a, b) \in R \times R \mid a + I = b + I\}.$$

It is easy to see that Ω is a congruence, since it is clearly an equivalence relation respecting addition and multiplication of R. In other words, a and b are congruent, exactly when they determine the same coset.

The next example moves us momentarily to semirings†.

Example 5.19. Let R be a layered semiring†. The relation $\cong_\nu$ given in Definition 3.23 is a congruence. This follows from Proposition 3.26.

Note that we can make R a semiring by adding the zero element, as described in the remark after Definition 3.1. Therefore we can assume that $\cong_\nu$ is defined also for a semiring.

The following proposition leads to a more practical way to describe congruences. It utilizes the direct product $R \times R$, where R is a semiring. Such a structure is clearly a semiring, when defining both addition and multiplication componentwise, based on the corresponding operations of R.

Proposition 5.20. *Let R be a semiring and Ω a relation on it. Then Ω is a congruence, if and only if it is such a subset of the direct product $R \times R$ that is both an equivalence relation on R and a subsemiring of $R \times R$.*

Proof. "$\Rightarrow$" As a congruence, Ω is an equivalence relation on R.

Let $(a, b), (c, d) \in \Omega$, for $a, b, c, d \in R$. Since Ω respects both addition and multiplication in the way given in Definition 5.17, we obtain directly

$$(a, b) + (c, d) = (a + c, b + d) \in \Omega,$$
$$(a, b) \cdot (c, d) = (a \cdot c, b \cdot d) \in \Omega.$$

Since Ω is reflexive, we have $(0, 0), (1, 1) \in \Omega$. These are the additive and multiplicative neutral elements in the direct product $R \times R$, and thus, we have now shown Ω to be also a subsemiring of $R \times R$.

"$\Leftarrow$" Suppose that $\Omega \subset R \times R$ is both an equivalence relation on R and a subsemiring of $R \times R$. The only task is to show that Ω respects addition and multiplication as required in Definition 5.17, but this follows from the assumption that Ω is a semiring. Namely, if $(a, b), (c, d) \in \Omega$, for $a, b, c, d \in R$, then

$$(a + c, b + d) = (a, b) + (c, d) \in \Omega,$$
$$(a \cdot c, b \cdot d) = (a, b) \cdot (c, d) \in \Omega,$$

which means that $a + c \equiv b + d$ and $a \cdot c \equiv b \cdot d$. □

Corollary 5.21. *Let $(R, +, \cdot)$ be a semiring, $a, b, c, d \in R$, and $\Omega \subset R \times R$ a relation. Then Ω is a congruence on R, if and only if the following conditions hold true.*

(i) *Ω is an equivalence relation.*

(ii) *If $(a, b), (c, d) \in \Omega$, then $(a + c, b + d) \in \Omega$.*

(iii) *If $(a, b), (c, d) \in \Omega$, then $(a \cdot c, b \cdot d) \in \Omega$.*

In the same way as with ideals, an intersection of congruences is a congruence, as will be shown in the following proposition.

Proposition 5.22. *Let R be a semiring and $(\Omega_i)_{i \in I}$ a family of congruences on R. Then*

$$\Omega := \bigcap_{i \in I} \Omega_i$$

is a congruence on R.

Proof. To prove reflexivity, suppose that $x \in R$. Each Ω_i is a congruence, and thus, $(x, x) \in \Omega_i$ for all $i \in I$. Therefore $(x, x) \in \Omega$.

To prove symmetry, suppose that $(x, y) \in \Omega$, for $x, y \in R$. This means that $(x, y) \in \Omega_i$ for all $i \in I$. Each Ω_i is a congruence, and thus, $(y, x) \in \Omega_i$ for all $i \in I$. Therefore $(y, x) \in \Omega$.

To prove transitivity, suppose that $(x, y), (y, z) \in \Omega$, for $x, y, z \in R$. This means that $(x, y), (y, z) \in \Omega_i$ for all $i \in I$. Each Ω_i is a congruence, and thus, $(x, z) \in \Omega_i$ for all $i \in I$. Therefore $(x, z) \in \Omega$.

To prove Ω to respect the operations of R, suppose that $(x, y), (x', y') \in \Omega$, for $x, y, x', y' \in R$. This means that $(x, y), (x', y') \in \Omega_i$ for all $i \in I$. Each Ω_i is a congruence, and thus, $(x + x', y + y'), (xx', yy') \in \Omega_i$ for all $i \in I$. Therefore $(x + x', y + y'), (xx', yy') \in \Omega$. □

As with ideals, we can consider congruences generated by a set (which is now a relation).

Definition 5.23. Let R be a semiring and Ω a relation on it. The *congruence generated by* Ω is the smallest congruence containing Ω. It is denoted as $\langle \Omega \rangle$.

As proved in Proposition 5.22, an intersection of congruences is a congruence. Therefore

$$\langle \Omega \rangle = \bigcap_{\Omega' \supset \Omega} \Omega',$$

where each Ω' is a congruence on R.

If $\Omega = \{(a, b)\}$, for $a, b \in R$, we denote $\langle (a, b) \rangle$ instead of $\langle \{(a, b)\} \rangle$.

As with ideals, we can consider quotient structures also with congruences.

Definition 5.24. Let R be a semiring and Ω a congruence on it. If $a \in R$, we denote

$$a/\Omega := \{b \in R \mid (a, b) \in \Omega\}$$

for the set of those elements that are congruent with a certain element.

Moreover, denote

$$R/\Omega := \{a/\Omega \mid a \in R\}$$

for the set of such sets, the elements of which are congruent with each other. The above set is called a *quotient semiring*.

Remark. A quotient semiring is indeed a semiring. Namely, suppose that $(R, +, \cdot)$ is a semiring and Ω a congruence on R. Define addition and multiplication operations in R/Ω as

$$a/\Omega + b/\Omega = (a + b)/\Omega \quad \text{and} \quad a/\Omega \cdot b/\Omega = (a \cdot b)/\Omega,$$

for all $a/\Omega, b/\Omega \in R/\Omega$, where $a, b \in R$. Moreover, if 0 is the additive neutral element in R and 1 is the multiplicative one, then $0/\Omega$ and $1/\Omega$ are those of R/Ω, respectively. With these assumptions, it is easy to see that R/Ω is a semiring.

Combining quotient structures with the following special kind of congruence lead to the isomorphism theorem of semirings.

Definition 5.25. Let R and S be semirings and $\varphi : R \to S$ a homomorphism. Define a relation

$$a \equiv b \iff \varphi(a) = \varphi(b),$$

for all $a, b \in R$. Then, $\equiv$ is clearly a congruence, called the *kernel congruence of* φ. It is denoted as $\operatorname{Ker}\varphi$.

Proposition 5.26. *Let R and S be semirings and $\varphi : R \to S$ a homomorphism. Then*

$$R/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi,$$

where $\operatorname{Ker}\varphi$ *is the kernel congruence given in Definition 5.25.*

Proof. By writing

$$\operatorname{Ker}\varphi = \{(a, b) \in R \times R \mid \varphi(a) = \varphi(b)\},$$

we can easily see the claim to hold true. Namely, those elements of R that have the same image in φ are congruent in $\operatorname{Ker}\varphi$. □

Remark. Especially, φ is injective, exactly when

$$\begin{aligned}\operatorname{Ker}\varphi &= \{(a, b) \in R \times R \mid \varphi(a) = \varphi(b)\} = \{(a, b) \in R \times R \mid a = b\} \\ &= \{(a, a) \mid a \in R\}.\end{aligned}$$

In other words, $\operatorname{Ker}\varphi$ is the smallest set of relations that still is an equivalence relation on R. This corresponds to the kernel of an injective ring homomorphism that is $\{0\}$, i.e. the smallest set that still is an ideal.

A congruence consisting only of reflexive relations is called a *trivial congruence.*

In addition to the (first) isomorphism theorem of semirings, as given in Proposition 5.26, we can prove the third (or second) isomorphism theorem of them, as follows.

Proposition 5.27. *Let R be a semiring with two congruences I and J such that $J \subset I$. Then*

$$I/J := \{(a/J, b/J) \in R/J \times R/J \mid a/I = b/I, \textit{ for } a, b \in R\}$$

is a congruence on R/J, and $(R/J)/(I/J) \cong R/I$.

Proof. Based on the above definition, it is easy to see that I/J is an equivalence relation. To show that I/J respects addition and multiplication of R/J, suppose that $(a/J, b/J), (c/J, d/J) \in I/J$, for $a, b, c, d \in R$. Therefore $a/I = b/I$ and $c/I = d/I$. By applying the addition and multiplication rules given in the remark after Definition 5.24, we obtain $(a + c)/I = (b + d)/I$ and $(ac)/I = (bd)/I$. Therefore $((a+c)/J, (b+d)/J), ((ac)/J, (bd)/J) \in I/J$. Hence, I/J respects addition and multiplication as required, and thus, we have until now proved it to be a congruence on R/J.

According to Definition 5.24, a quotient semiring is constructed by a semiring modulo a congruence on it. Therefore R/J is a semiring, and as just proved, I/J a congruence on it, when also $(R/J)/(I/J)$ is a semiring. We define now a composition map between these semirings, as follows

$$\begin{array}{ccccc} R & \xrightarrow{\varphi} & R/J & \xrightarrow{\psi} & (R/J)/(I/J), \\ a & \mapsto & a/J & \mapsto & (a/J)/(I/J). \end{array}$$

Each component map is a map from a semiring to its quotient semiring. Such a map is clearly well-defined. Furthermore, it is a homomorphism, since

$$\begin{aligned} \varphi(a + b) &= (a + b)/J = a/J + b/J = \varphi(a) + \varphi(b), \\ \varphi(a \cdot b) &= (a \cdot b)/J = a/J \cdot b/J = \varphi(a) \cdot \varphi(b), \end{aligned}$$

for all $a, b \in R$, and clearly also $\varphi(0) = 0/J$ and $\varphi(1) = 1/J$. Since also ψ is a map from a semiring to its quotient semiring, the same kind of conclusion is valid for it, too. Therefore both φ and ψ are homomorphisms. The surjective property of the maps is obvious.

Moreover,

$$\begin{aligned} \mathrm{Ker}(\psi \circ \varphi) &= \{(a, b) \in R \times R \mid \psi(\varphi(a)) = \psi(\varphi(b))\} \\ &= \{(a, b) \in R \times R \mid (a/J)/(I/J) = (b/J)/(I/J)\} \\ &= \{(a, b) \in R \times R \mid (a/J, b/J) \in I/J\} \\ &= \{(a, b) \in R \times R \mid a/I = b/I\} \\ &= \{(a, b) \in R \times R \mid (a, b) \in I\} \\ &= I, \end{aligned}$$

and thus, Proposition 5.26 implies $(R/J)/(I/J) \cong R/I$. $\square$

5.4 Advanced properties of congruences

In the previous section, we have seen that congruences have many properties in common with ideals. This section describes additional properties, the connection of which to ideals is even stronger than those presented in the previous section. However, this requires a special operation for congruences, called twisted product [1, p. 5]. It provides a more natural correspondence between ideals and congruences than does the usual product, given as property (iii) in Corollary 5.21. On the other hand, it requires the existence of zero.

Definition 5.28. Let $(R, +, \cdot)$ be a semiring and $\Omega \subset R \times R$ a congruence. The *twisted product* on Ω is defined as

$$(a, b) \star (c, d) = (ac + bd, ad + bc),$$

for all $(a, b), (c, d) \in \Omega$, where $a, b, c, d \in R$.

Remark. To distinguish twisted product from Cartesian product, we use a different symbol for it, contrary to [1].

The element $(1, 0)$ is the neutral element of twisted product. Namely,

$$(a, b) \star (1, 0) = (a, b) = (1, 0) \star (a, b),$$

for all $(a, b) \in \Omega$, where $a, b \in R$.

Twisted product fit together with addition, as shown in the following lemma.

Lemma 5.29. *Twisted product distributes over the addition of a semiring.*

Proof. Let $(R, +, \cdot)$ be a semiring and Ω a congruence on R. Suppose that $(a, b), (c, d), (e, f) \in \Omega$, for $a, b, c, d, e, f \in R$. Then

$$\begin{aligned}
(a, b) \star ((c, d) + (e, f)) &= (a, b) \star (c + e, d + f) \\
&= (ac + ae + bd + bf, ad + af + bc + be) \\
&= (ac + bd, ad + bc) + (ae + bf, af + be) \\
&= (a, b) \star (c, d) + (a, b) \star (e, f),
\end{aligned}$$

and

$$\begin{aligned}
((a, b) + (c, d)) \star (e, f) &= (a + c, b + d) \star (e, f) \\
&= (ae + ce + bf + df, af + cf + be + de) \\
&= (ae + bf, af + be) + (ce + df, cf + de) \\
&= (a, b) \star (e, f) + (c, d) \star (e, f).
\end{aligned}$$

□

Based on twisted product, we can provide an alternative condition for property (iii) in Corollary 5.21. The equivalence between these properties is proved in the following proposition.

Proposition 5.30. *Let R be a semiring, $a, b, c, d \in R$, and $\Omega \subset R \times R$ an equivalence relation that respects addition of R, i.e. Ω satisfies the conditions (i) and (ii) in Corollary 5.21. The following conditions are equivalent:*

(i) If $(a,b),(c,d) \in \Omega$, then $(ac,bd) \in \Omega$.

(ii) If $(a,b) \in \Omega$, then $(c,d) \star (a,b) \in \Omega$, for all $c,d \in R$.

Proof. "(i) $\Rightarrow$ (ii)" Suppose that $(a,b) \in \Omega$ and $c,d \in R$. As an equivalence relation, Ω is reflexive, and thus, we have $(c,c),(d,d) \in \Omega$. Now, (i) implies $(ca,cb),(da,db) \in \Omega$. As an equivalence relation, Ω is symmetric, and thus, we also have $(db,da) \in \Omega$. Since Ω respects addition of R, we obtain

$$(c,d) \star (a,b) = (ca+db, cb+da) \in \Omega.$$

This holds true for all $c,d \in R$, since these elements were chosen arbitrarily.

"(ii) $\Rightarrow$ (i)" Suppose that $(a,b),(c,d) \in \Omega$. Now $0 \in R$, and thus, (ii) implies both

$$\begin{aligned}(ac,cb) &= (c,0) \star (a,b) \in \Omega,\\ (cb,bd) &= (b,0) \star (c,d) \in \Omega.\end{aligned}$$

As an equivalence relation, Ω is transitive, and thus, $(ac,bd) \in \Omega$. $\square$

Remark. The proof of the latter implication requires the existence of zero in the semiring. Therefore it cannot be applied to a semiring† as such. However, a congruence on a semiring† satisfies condition (ii) in the above proposition.

As shown in the remark after Definition 5.28, $(1,0)$ is the neutral element of twisted product. Twisted product makes this element act in congruences in the same way as the unit element in ideals of rings, as will be shown in the following lemma.

Lemma 5.31. *Let R be a semiring and $\Omega \subset R \times R$ a congruence. Then Ω is a proper congruence, if and only if $(1,0) \notin \Omega$.*

Proof. If $(1,0) \notin \Omega$, then clearly Ω is proper. If $(1,0) \in \Omega$, then based on Proposition 5.30,

$$(a,b) = (a,b) \star (1,0) \in \Omega,$$

for all $a,b \in R$. Therefore $\Omega = R \times R$. $\square$

Remark. Based on Proposition 5.20, a congruence equipped with componentwise addition and multiplication, with $(0,0)$ and $(1,1)$ as the neutral elements of these operations, is a subsemiring of $R \times R$. However, a congruence equipped with a twisted product is more like an ideal. This follows from condition (ii) in Proposition 5.30, which is a very ideal-like property.

We will next define the sum and product of congruences. The product is based on twisted product, and the sum can be proved to be a congruence based on twisted product.

Definition 5.32. Let R be a semiring, and $\Omega, \Omega' \subset R \times R$ congruences. The *(twisted) product of congruences* is the generated congruence

$$\Omega \star \Omega' := \langle (a,b) \star (a',b') \mid (a,b) \in \Omega, (a',b') \in \Omega' \rangle.$$

Definition 5.33. Let R be a semiring, and $(\Omega_i)_{i \in I}$ a family of congruences on R. The *sum of congruences* is the set

$$\sum_{i \in I} \Omega_i := \{\sum_{i \in I} (a_i, b_i) \mid (a_i, b_i) \in \Omega_i, \text{ for all } i \in I\},$$

such that the set

$$\{i \in I \mid (a_i, b_i) \neq (0,0)\}$$

is finite.

The following lemma shows that the sum of congruences is indeed a congruence.

Lemma 5.34. *Let R be a semiring, and $(\Omega_i)_{i \in I}$ a family of congruences on R. The sum $\sum_{i \in I} \Omega_i$ is a congruence.*

Proof. Suppose that $(a_i, b_i) \in \Omega_i$, for some $i \in I$. Clearly $(0,0) \in \Omega_i$ for all $i \in I$, and thus,

$$(a_i, b_i) = (0,0) + \cdots + (0,0) + (a_i, b_i) + (0,0) + \cdots \in \sum_{i \in I} \Omega_i.$$

This proves that $\Omega_i \subset \sum_{i \in I} \Omega_i$, which implies further that $\sum_{i \in I} \Omega_i$ is reflexive.

Symmetry and transitive properties follow easily from those properties of all Ω_i. Similarly, $\sum_{i \in I} \Omega_i$ can be proved to respect addition by reflecting to the corresponding property of all Ω_i.

Twisted product can be applied to show $\sum_{i \in I} \Omega_i$ to respect multiplication. Suppose that $c, d \in R$, and $\sum_{i \in I} (a_i, b_i) \in \sum_{i \in I} \Omega_i$, for $(a_i, b_i) \in \Omega_i$ for all $i \in I$. Based on Lemma 5.29, twisted product distributes over addition, and thus,

$$(\sum_{i \in I} (a_i, b_i)) \star (c,d) = \sum_{i \in I} \underbrace{((a_i, b_i) \star (c,d))}_{\in \Omega_i} \in \sum_{i \in I} \Omega_i.$$

Namely, each $(a_i, b_i) \star (c,d)$ is an element of Ω_i based on Proposition 5.30, since each Ω_i is a congruence. Therefore, the whole expression above proves that $\sum_{i \in I} \Omega_i$ satisfies condition (ii) in Proposition 5.30. On the other hand, we have shown above that $\sum_{i \in I} \Omega_i$ is an equivalence relation respecting addition. Hence, Proposition 5.30 implies also condition (i) to hold, i.e. $\sum_{i \in I} \Omega_i$ respects multiplication. $\square$

Remark. Since we above apply Proposition 5.30 in the direction of the conditions from (ii) to (i), the assumption that R is a semiring (with zero) is totally necessary. However, later in the context of congruence varieties, the sum of congruences is not obligate.

By applying twisted product, we obtain a result that is similar to the case of ideals.

Lemma 5.35. *Let R be a semiring and $\Omega, \Omega' \subset R \times R$ congruences. Then $\Omega \star \Omega' \subset \Omega$.*

Proof. The claim follows directly from Definition 5.32 and property (ii) in Proposition 5.30. Namely, an arbitrary element of $\Omega \star \Omega'$ is of the form $(a, b) \star (a', b')$, for $(a, b) \in \Omega$ and $(a', b') \in \Omega'$, where $a, b, a', b' \in R$. Since $(a, b) \in \Omega$, Proposition 5.30 implies $(a, b) \star (a', b') \in \Omega$. $\square$

Remark. The above kind of subset relation holds true for twisted product but not for the normal product of congruences. To see this, replace twisted product in Definition 5.32 with the normal product as

$$\Omega \cdot \Omega' := \langle (a \cdot a', b \cdot b') \mid (a, b) \in \Omega, (a', b') \in \Omega' \rangle.$$

Now, it does not hold $\Omega \cdot \Omega' \subset \Omega$. (As an example, consider the case, where Ω is trivial and Ω' is not.) Instead we obtain such a strange result that $\Omega \subset \Omega \cdot \Omega'$. Namely, each relation $(a, b) \in \Omega$ can be written in the form $(a \cdot 1, b \cdot 1)$, and $(1, 1)$ is an element of any congruence, and thus, $(1, 1) \in \Omega'$. Therefore $(a, b) \in \Omega \cdot \Omega'$.

As shown above, twisted product has more natural consequences than the normal product of congruences. The next lemma reveals how trivial congruence corresponds to a zero ideal, when applying twisted product.

Lemma 5.36. *Let R be a totally ordered cancellative semiring, and $\Omega, \Omega' \subset R \times R$ congruences. Then $\Omega \star \Omega'$ is trivial, if and only if Ω or Ω' is trivial.*

Proof. Suppose first that Ω is trivial. This means that all its elements are of the form (a, a), for $a \in R$. Therefore

$$(a, a) \star (a', b') = (aa' + ab', ab' + aa') \in \Omega \star \Omega',$$

for all $(a', b') \in \Omega'$, where a', b' in R. Hence $\Omega \star \Omega'$ is trivial.

Suppose next that $\Omega \star \Omega'$ is trivial. This means that $aa' + bb' = ab' + ba'$ for all $(a, b) \in \Omega$ and $(a', b') \in \Omega$, where $a, a', b, b' \in R$. Suppose against the claim that $a \neq b$ and $a' \neq b'$. Since R is totally ordered, we can assume, for example, that $a < b$ and $a' < b'$. These inequations imply $b > 0$ and $b' > 0$, respectively, since zero is the smallest element in a totally ordered semiring, as discussed in the remark after Example 3.13. Since R is cancellative, Proposition 2.26 (and the remark after it) implies that multiplication with a non-zero element preserves the strict order. Therefore, we obtain $ab' < bb'$ and $ba' < bb'$, which imply together $ab' + ba' < bb'$. Since $aa' + bb' = ab' + ba'$, the previous inequation can be written as $aa' + bb' < bb'$, which is a contradiction. Hence, it must be $a = b$ or $a' = b'$, which means that Ω or Ω' is trivial. $\square$

5.5 Generated congruences

Recall that a generated ideal of a ring can be expressed as a linear combination of its generators. A corresponding result in the case of congruences can be achieved by the following two propositions. The first one is corrected from [1, p. 6] (Proposition 2.8), and the second one taken from [1, p. 7] (Corollary 2.9) with a different (corrected) proof.

Proposition 5.37. *Let R be a semiring and $S \subset R \times R$ a reflexive and symmetric relation that respects addition and multiplication, i.e. satisfies the properties (ii) and (iii) in Corollary 5.21. The set*

$$T := \{(a_1, a_k) \in R \times R \,|(a_1, a_2), (a_2, a_3), \ldots, (a_{k-1}, a_k) \in S, \\ \textit{for } a_2, \ldots, a_{k-1} \in R, \ \textit{and } k \geq 2\}$$

is the transitive closure of S, i.e. $\langle S \rangle = T$.

Proof. "$\subset$" By setting $k = 2$, we can see that $S \subset T$.

We will next prove T to be a congruence. Reflexive property follows from the subset relation $S \subset T$. Symmetric property follows from that of S, as follows. Suppose that $(a_1, a_k) \in T$, for $a_1, a_k \in R$. Therefore $(a_1, a_2), \ldots, (a_{k-1}, a_k) \in S$, for $a_2, \ldots, a_{k-1} \in R$ and $k \geq 2$.. Now, S is symmetric, and thus, $(a_k, a_{k-1}), \ldots, (a_2, a_1) \in S$. Hence, $(a_k, a_1) \in T$.

To show transitive property to hold, suppose that $(a_1, a_j), (a_j, a_k) \in T$, for $a_1, a_j, a_k \in R$. Therefore

$$(a_1, a_2), \ldots, (a_{j-1}, a_j), (a_j, a_{j+1}), \ldots, (a_{k-1}, a_k) \in S,$$

for $a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{k-1} \in R$, where $j \geq 2$ and $k \geq j+1$. By combining the inequations, we have $k \geq 3$. Therefore $(a_1, a_2), \ldots, (a_{k-1}, a_k) \in S$, and thus, $(a_1, a_k) \in T$.

To show T to respect addition, suppose that $(a_1, a_k), (b_1, b_j) \in T$, for some $a_1, a_k, b_1, b_j \in R$. Without loss of generality, we can assume that $j \leq k$. Now, all the elements

$$(a_1, a_2), \ldots, (a_{j-2}, a_{j-1}), (a_{j-1}, a_j), (a_j, a_{j+1}), \ldots (a_{k-2}, a_{k-1}), (a_{k-1}, a_k), \\ (b_1, b_2), \ldots, (b_{j-2}, b_{j-1}), \underbrace{(b_{j-1}, b_{j-1}), (b_{j-1}, b_{j-1}), \ldots, (b_{j-1}, b_{j-1})}_{k-j \text{ pieces}}, (b_{j-1}, b_j),$$

for $a_2, \ldots, a_{j-2}, a_{j-1}, a_j, a_{j+1}, \ldots, a_{k-2}, a_{k-1}, b_2, \ldots, b_{j-2}, b_{j-1} \in R$ and $k \geq j \geq 2$, are elements of S. Since S respects addition, it includes all the sums

$$(a_1 + b_1, a_2 + b_2), \ldots, (a_{j-2} + b_{j-2}, a_{j-1} + b_{j-1}), \\ (a_{j-1} + b_{j-1}, a_j + b_{j-1}), \\ (a_j + b_{j-1}, a_{j+1} + b_{j-1}), \ldots, (a_{k-2} + b_{j-1}, a_{k-1} + b_{j-1}), \\ (a_{k-1} + b_{j-1}, a_k + b_j).$$

Therefore $(a_1+b_1, a_k+b_j) \in T$. By replacing the operators of addition with those of multiplication, we can see T to respect multiplication, as well.

By definition, $\langle S\rangle$ is the smallest congruence containing S. We have shown above that T is a congruence containing S. Therefore $\langle S\rangle \subset T$.

"$\supset$" Suppose that $(a_1, a_k) \in T$, for $a_1, a_k \in R$. Therefore

$$(a_1, a_2), \ldots, (a_{k-1}, a_k) \in S \subset \langle S\rangle,$$

for $a_2, \ldots, a_{k-1} \in R$ and $k \geq 2$. As a congruence, $\langle S\rangle$ is a transitive, and thus, $(a_1, a_k) \in \langle S\rangle$. □

Remark. In [1] (Proposition 2.8), the set of transitive closure was defined as

$$T' := \{(a_1, a_3) \in R \times R \mid (a_1, a_2), (a_2, a_3) \in S, \text{ for } a_2 \in R\},$$

but the problem is that the above set is not necessarily transitive. Namely T' is the same set as T with the value $k = 3$. However, if S includes a chain of elements, $(a_1, a_2), (a_2, a_3), \ldots, (a_{k-1}, a_k)$, such that $k > 3$, then T' is not necessarily transitive.

The next proposition exploits the denotation: If R is a semiring and $(a_i, b_i)_{i\in I}$ a family of elements of $R \times R$, we write

$$\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}} = \prod_{i\in I} a_i^{m_i} b_i^{n_i},$$

for $\boldsymbol{m} = (\ldots, m_i, \ldots), \boldsymbol{n} = (\ldots, n_i, \ldots) \in \mathbb{N}^{(I)}$, where only finitely many of the exponents (m_i and n_i) differ from zero.

Proposition 5.38. *Let R be a semiring, and Ω a generated congruence on R, i.e. $\Omega := \langle (a_i, b_i) \mid a_i, b_i \in R, \text{ for } i \in I\rangle$. The elements in Ω are precisely those of the form*

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}}, \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} s_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}}\Big), \tag{5.1}$$

where, for all $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^{(I)}$, $r_{\boldsymbol{m},\boldsymbol{n}}, s_{\boldsymbol{m},\boldsymbol{n}} \in R$, all but finitely many of which are zero, and

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{m}} = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} s_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{m}}.$$

(The previous equation is called an equality condition*).*

Proof. Let T denote the set in the claim, i.e. the set consisting of the elements of the form given in (5.1). Let $S \subset R \times R$ denote the set, the elements of which are of the form

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}}, \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{m}}\Big),$$

where, for all $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^{(I)}$, $r_{\boldsymbol{m},\boldsymbol{n}} \in R$, all but finitely many of which are zero. We will show that $T \subset \langle S \rangle \subset \Omega \subset T$.

"$T \subset \langle S \rangle$":

Our aim is to apply Proposition 5.37. For this purpose, we will first prove S to satisfy the other requirements of a congruence except for the transitive property. By choosing $r_{\boldsymbol{m},\boldsymbol{n}} = 0$, for all non-zero $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^{(I)}$, we can see that S is reflexive. By changing the roles of $\boldsymbol{m}$ and $\boldsymbol{n}$, we can see that S is symmetric. Moreover, the sums and products of the elements of S are again elements of S. In sums, the coefficients are of the form $r_{\boldsymbol{m},\boldsymbol{n}} + s_{\boldsymbol{m},\boldsymbol{n}}$, which are elements of R, if both the summand coefficients are such. In products, the multipliers are of the form

$$r_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{m}} \boldsymbol{b}^{\boldsymbol{n}} \quad \text{and} \quad s_{\boldsymbol{m'},\boldsymbol{n'}} \boldsymbol{a}^{\boldsymbol{m'}} \boldsymbol{b}^{\boldsymbol{n'}},$$

where $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{m'}, \boldsymbol{n'} \in \mathbb{N}^{(I)}$ and $r_{\boldsymbol{m},\boldsymbol{n}}, s_{\boldsymbol{m'},\boldsymbol{n'}} \in R$, and thus,

$$r_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{m}} \boldsymbol{b}^{\boldsymbol{n}} \cdot s_{\boldsymbol{m'},\boldsymbol{n'}} \boldsymbol{a}^{\boldsymbol{m'}} \boldsymbol{b}^{\boldsymbol{n'}} = r_{\boldsymbol{m},\boldsymbol{n}} s_{\boldsymbol{m'},\boldsymbol{n'}} \boldsymbol{a}^{\boldsymbol{m}+\boldsymbol{m'}} \boldsymbol{b}^{\boldsymbol{n}+\boldsymbol{n'}}.$$

Therefore, if $r_{\boldsymbol{m},\boldsymbol{n}}, s_{\boldsymbol{m},\boldsymbol{n}} \in R$, all but finitely many of which are zero, then the product between

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{m}} \boldsymbol{b}^{\boldsymbol{n}}, \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{n}} \boldsymbol{b}^{\boldsymbol{m}} \Big)$$

and

$$\Big(\sum_{\boldsymbol{m'},\boldsymbol{n'} \in \mathbb{N}^{(I)}} s_{\boldsymbol{m'},\boldsymbol{n'}} \boldsymbol{a}^{\boldsymbol{m'}} \boldsymbol{b}^{\boldsymbol{n'}}, \sum_{\boldsymbol{m'},\boldsymbol{n'} \in \mathbb{N}^{(I)}} s_{\boldsymbol{m'},\boldsymbol{n'}} \boldsymbol{a}^{\boldsymbol{n'}} \boldsymbol{b}^{\boldsymbol{m'}} \Big)$$

is

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{m'},\boldsymbol{n'} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} s_{\boldsymbol{m'},\boldsymbol{n'}} \boldsymbol{a}^{\boldsymbol{m}+\boldsymbol{m'}} \boldsymbol{b}^{\boldsymbol{n}+\boldsymbol{n'}}, \sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{m'},\boldsymbol{n'} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} s_{\boldsymbol{m'},\boldsymbol{n'}} \boldsymbol{a}^{\boldsymbol{n}+\boldsymbol{n'}} \boldsymbol{b}^{\boldsymbol{m}+\boldsymbol{m'}} \Big),$$

which is of the desired form.

We have now proved S to be a reflexive and symmetric relation on R, respecting addition and multiplication. Therefore Proposition 5.37 implies

$$\begin{aligned} \langle S \rangle = \{(a_1, a_k) \in R \times R \,|(a_1, a_2), \ldots, (a_{k-1}, a_k) \in S, \\ \text{for } a_2, \ldots, a_{k-1} \in R, \text{ and } k \geq 2\}. \end{aligned}$$

We will next pay attention to T. Write

$$\begin{aligned} r_1 &:= \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{m}} \boldsymbol{b}^{\boldsymbol{n}}, \quad s_1 := \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} s_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{m}} \boldsymbol{b}^{\boldsymbol{n}}, \\ r_2 &:= \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{n}} \boldsymbol{b}^{\boldsymbol{m}}, \quad s_2 := \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} s_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{n}} \boldsymbol{b}^{\boldsymbol{m}}, \end{aligned}$$

where, for all $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^{(I)}$, $r_{\boldsymbol{m},\boldsymbol{n}}, s_{\boldsymbol{m},\boldsymbol{n}} \in R$, all but finitely many of which are zero. The definition of T can now be written as

$$T = \{(r_1, s_1) \mid r_2 = s_2\}.$$

Since $(r_1, r_2), (s_1, s_2) \in S$, we can write further

$$T = \{(r_1, s_1) \mid (r_1, r_2), (s_2, s_1) \in S \text{ and } r_2 = s_2\}.$$

By comparing the definitions of T and $\langle S \rangle$, we can see that $T \subset \langle S \rangle$. Namely, T is of the form of $\langle S \rangle$ on condition $k = 3$.

"$\langle S \rangle \subset \Omega$":

We start by proving $S \subset \Omega$. Clearly, $(a_i, b_i) \in \Omega$ for all $i \in I$. Since Ω is a congruence, it respects multiplication, and thus, we obtain

$$\Big(\prod_{i \in I} a_i^{m_i}, \prod_{i \in I} b_i^{m_i}\Big) \in \Omega,$$

where $m_i \in \mathbb{N}$ for all $i \in I$. As a congruence, Ω is symmetric, and thus, we also have

$$\Big(\prod_{i \in I} b_i^{n_i}, \prod_{i \in I} a_i^{n_i}\Big) \in \Omega.$$

where $n_i \in \mathbb{N}$ for all $i \in I$. Furthermore, multiplication between two previous elements gives

$$\Big(\prod_{i \in I} a_i^{m_i} b_i^{n_i}, \prod_{i \in I} a_i^{n_i} b_i^{m_i}\Big) \in \Omega.$$

As a congruence, Ω is reflexive, and thus, $(r_{\boldsymbol{m},\boldsymbol{n}}, r_{\boldsymbol{m},\boldsymbol{n}}) \in \Omega$, for all $r_{\boldsymbol{m},\boldsymbol{n}} \in R$. Since Ω respects addition, we obtain

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} \prod_{i \in I} a_i^{m_i} b_i^{n_i}, \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} \prod_{i \in I} a_i^{n_i} b_i^{m_i}\Big) \in \Omega,$$

where $\boldsymbol{m} = (\ldots, m_i, \ldots), \boldsymbol{n} = (\ldots, n_i, \ldots) \in \mathbb{N}^{(I)}$ such that only finitely many of the exponents (m_i and n_i) differ from zero, and only finitely many of the coefficients $r_{\boldsymbol{m},\boldsymbol{n}}$ differ from zero. This can be written in a shorter way, as

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{m}} \boldsymbol{b}^{\boldsymbol{n}}, \sum_{\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}} \boldsymbol{a}^{\boldsymbol{n}} \boldsymbol{b}^{\boldsymbol{m}}\Big) \in \Omega,$$

which proves that $S \subset \Omega$. Therefore Ω is a congruence containing S. On the other hand, $\langle S \rangle$ is the smallest congruence containing S, and thus, $\langle S \rangle \subset \Omega$.

"$\Omega \subset T$":

We will show T to be a congruence, containing the elements (a_i, b_i) for all $i \in I$. By choosing $r_{\boldsymbol{m},\boldsymbol{n}} = 0$, for all non-zero $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^{(I)}$, we can see that T is reflexive. Clearly, T is symmetric. Transitivity can be seen as follows. Suppose that $(r, s), (s, t) \in T$ with equality conditions $r' = s'$ and $s' = t'$, respectively. The equality conditions imply $r' = t'$, and thus, $(r, t) \in T$.

To show T to respect addition and multiplication of R, suppose that

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}}, \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} s_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}}\Big),$$
$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} t_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}}, \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} u_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}}\Big) \in T,$$

where, for all $\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}$, $r_{\boldsymbol{m},\boldsymbol{n}}, s_{\boldsymbol{m},\boldsymbol{n}}, t_{\boldsymbol{m},\boldsymbol{n}}, u_{\boldsymbol{m},\boldsymbol{n}} \in R$, all but finitely many of which are zero. This requires the equations

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{m}} = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} s_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{m}},$$
$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} t_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{m}} = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} u_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{m}}$$

to hold. By adding together both sides, we obtain

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} (r_{\boldsymbol{m},\boldsymbol{n}}+t_{\boldsymbol{m},\boldsymbol{n}})\boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{m}} = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} (s_{\boldsymbol{m},\boldsymbol{n}}+u_{\boldsymbol{m},\boldsymbol{n}})\boldsymbol{a}^{\boldsymbol{n}}\boldsymbol{b}^{\boldsymbol{m}},$$

which proves that the sum of the aforementioned elements is an element of T. Consider next multiplication of the same elements. We multiply terms as

$$r_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}} \quad\text{and}\quad t_{\boldsymbol{m}',\boldsymbol{n}'}\boldsymbol{a}^{\boldsymbol{m}'}\boldsymbol{b}^{\boldsymbol{n}'},$$

which gives

$$r_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{a}^{\boldsymbol{m}}\boldsymbol{b}^{\boldsymbol{n}} \cdot t_{\boldsymbol{m}',\boldsymbol{n}'}\boldsymbol{a}^{\boldsymbol{m}'}\boldsymbol{b}^{\boldsymbol{n}'} = r_{\boldsymbol{m},\boldsymbol{n}}t_{\boldsymbol{m}',\boldsymbol{n}'}\boldsymbol{a}^{\boldsymbol{m}+\boldsymbol{m}'}\boldsymbol{b}^{\boldsymbol{n}+\boldsymbol{n}'}.$$

On the other hand, multiplying the equality conditions together side by side yields a new equality condition as

$$\sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{m}',\boldsymbol{n}'\in\mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}}t_{\boldsymbol{m}',\boldsymbol{n}'}\boldsymbol{a}^{\boldsymbol{n}+\boldsymbol{n}'}\boldsymbol{b}^{\boldsymbol{m}+\boldsymbol{m}'} = \sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{m}',\boldsymbol{n}'\in\mathbb{N}^{(I)}} s_{\boldsymbol{m},\boldsymbol{n}}u_{\boldsymbol{m}',\boldsymbol{n}'}\boldsymbol{a}^{\boldsymbol{n}+\boldsymbol{n}'}\boldsymbol{b}^{\boldsymbol{m}+\boldsymbol{m}'},$$

which proves that the product

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{m}',\boldsymbol{n}'\in\mathbb{N}^{(I)}} r_{\boldsymbol{m},\boldsymbol{n}}t_{\boldsymbol{m}',\boldsymbol{n}'}\boldsymbol{a}^{\boldsymbol{m}+\boldsymbol{m}'}\boldsymbol{b}^{\boldsymbol{n}+\boldsymbol{n}'}, \sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{m}',\boldsymbol{n}'\in\mathbb{N}^{(I)}} s_{\boldsymbol{m},\boldsymbol{n}}u_{\boldsymbol{m}',\boldsymbol{n}'}\boldsymbol{a}^{\boldsymbol{m}+\boldsymbol{m}'}\boldsymbol{b}^{\boldsymbol{n}+\boldsymbol{n}'}\Big)$$

is an element of T.

The final task is to show that $(a_i, b_i) \in T$, for all $i \in I$. This can be done by first choosing $r_{\boldsymbol{m},\boldsymbol{n}} = s_{\boldsymbol{m},\boldsymbol{n}} = 0_R$, for all $\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}$, except for two pairs of indices, which are

$$\boldsymbol{m}' = (0,\dots,0,\underset{i}{1},0,\dots) \quad\text{and}\quad \boldsymbol{m}'' = \boldsymbol{n}' = \boldsymbol{n}'' = (0,\dots),$$

and by setting the coefficients of these indices as

$$r_{\boldsymbol{m}',\boldsymbol{n}'} = 1_R, \quad r_{\boldsymbol{m}'',\boldsymbol{n}''} = 0_R, \quad s_{\boldsymbol{m}',\boldsymbol{n}'} = 0_R, \quad s_{\boldsymbol{m}'',\boldsymbol{n}''} = b_i.$$

These selections give

$$\begin{aligned}
&(a_i, b_i)\\
&= (a_i + 0_R, 0_R + b_i)\\
&= (1_R \cdot a_1^0 \cdots a_{i-1}^0 a_i^1 a_{i+1}^0 \cdots b_1^0 \cdots b_i^0 \cdots + 0_R \cdot a_1^0 \cdots a_i^0 \cdots b_1^0 \cdots b_i^0 \cdots ,\\
&\quad\ 0_R \cdot a_1^0 \cdots a_{i-1}^0 a_i^1 a_{i+1}^0 \cdots b_1^0 \cdots b_i^0 \cdots + b_i \cdot a_1^0 \cdots a_i^0 \cdots b_1^0 \cdots b_i^0 \cdots)\\
&= (r_{\boldsymbol{m}',\boldsymbol{n}'} \boldsymbol{a}^{\boldsymbol{m}'} \boldsymbol{b}^{\boldsymbol{n}'} + r_{\boldsymbol{m}'',\boldsymbol{n}''} \boldsymbol{a}^{\boldsymbol{m}''} \boldsymbol{b}^{\boldsymbol{n}''}, s_{\boldsymbol{m}',\boldsymbol{n}'} \boldsymbol{a}^{\boldsymbol{m}'} \boldsymbol{b}^{\boldsymbol{n}'} + s_{\boldsymbol{m}'',\boldsymbol{n}''} \boldsymbol{a}^{\boldsymbol{m}''} \boldsymbol{b}^{\boldsymbol{n}''})
\end{aligned}$$

and

$$\begin{aligned}
&r_{\boldsymbol{m}',\boldsymbol{n}'} \boldsymbol{a}^{\boldsymbol{n}'} \boldsymbol{b}^{\boldsymbol{m}'} + r_{\boldsymbol{m}'',\boldsymbol{n}''} \boldsymbol{a}^{\boldsymbol{n}''} \boldsymbol{b}^{\boldsymbol{m}''}\\
&\quad = 1_R \cdot a_1^0 \cdots a_i^0 \cdots b_1^0 \cdots b_{i-1}^0 b_i^1 b_{i+1}^0 \cdots + 0_R \cdot a_1^0 \cdots a_i^0 \cdots b_1^0 \cdots b_i^0 \cdots\\
&\quad = b_i + 0_R\\
&\quad = 0_R + b_i\\
&\quad = 0_R \cdot a_1^0 \cdots a_i^0 \cdots b_1^0 \cdots b_{i-1}^0 b_i^1 b_{i+1}^0 \cdots + b_i \cdot a_1^0 \cdots a_i^0 \cdots b_1^0 \cdots b_i^0 \cdots\\
&\quad = s_{\boldsymbol{m}',\boldsymbol{n}'} \boldsymbol{a}^{\boldsymbol{n}'} \boldsymbol{b}^{\boldsymbol{m}'} + s_{\boldsymbol{m}'',\boldsymbol{n}''} \boldsymbol{a}^{\boldsymbol{n}''} \boldsymbol{b}^{\boldsymbol{m}''}.
\end{aligned}$$

This proves that each element (a_i, b_i) is of the form required from the elements of T, and thus, $(a_i, b_i) \in T$ for all $i \in I$. Since T is a congruence containing elements (a_i, b_i) for all $i \in I$, we obtain $\Omega = \langle (a_i, b_i) \rangle \subset T$. □

Remark. The representation of the elements of T is not unique, and thus, a candidate element of T may have several equality conditions, some of which hold true and some others do not. If there exist a true equality condition for a candidate element, then the candidate element is an element of T.

Example 5.39. Let R be a semiring. Naturally, the generated congruence $\langle (a, a) \rangle$ is trivial for all $a \in R$. This can be seen based on Proposition 5.38 as follows. The elements of $\langle (a, a) \rangle$ are of the form

$$\Big(\sum_{m,n \in \mathbb{N}} r_{m,n} a^m a^n, \sum_{m,n \in \mathbb{N}} s_{m,n} a^m a^n \Big) = \Big(\sum_{m,n \in \mathbb{N}} r_{m,n} a^{m+n}, \sum_{m,n \in \mathbb{N}} s_{m,n} a^{m+n} \Big),$$

where, for all $m, n \in \mathbb{N}$, $r_{m,n}, s_{m,n} \in R$, all but finitely many of which are zero, and

$$\sum_{m,n \in \mathbb{N}} r_{m,n} a^{m+n} = \sum_{m,n \in \mathbb{N}} r_{m,n} a^n a^m = \sum_{m,n \in \mathbb{N}} s_{m,n} a^n a^m = \sum_{m,n \in \mathbb{N}} s_{m,n} a^{m+n}.$$

Therefore all the pairs of $\langle (a, a) \rangle$ are reflexive. On the other hand, $\langle (a, a) \rangle$ comprises all the reflexive pairs of $R \times R$, which can be seen by choosing $r_{m,n} = 0$, for all non-zero $m, n \in \mathbb{N}$.

Remark. If $(0_R)^0 = 1_R$, then the above result holds true, even if $a = 0_R$. Namely,

$$\Big(\sum_{m,n \in \mathbb{N}} r_{m,n} 0^m 0^n, \sum_{m,n \in \mathbb{N}} s_{m,n} 0^m 0^n \Big) = (r_{0,0}, s_{0,0}),$$

where $r_{m,n}, s_{m,n} \in R$ for all $m, n \in \mathbb{N}$, and

$$r_{0,0} = \sum_{m,n\in\mathbb{N}} r_{m,n}0^n0^m = \sum_{m,n\in\mathbb{N}} s_{m,n}0^n0^m = s_{0,0}.$$

Example 5.40. Let R be a semiring, and $\Omega := \langle (X, 0)\rangle$ a congruence on $R[X]$. Clearly, $(X, 0) \in \Omega$, and since congruences respect addition and multiplication, we have

$$(\sum_{m=1}^{\infty} r_m X^m, 0) \in \Omega,$$

where, for all $m \geq 1$, $r_m \in R$, all but finitely many of which are zero. In other words, each polynomial in $R[X]$ with no constant term is congruent with zero, and thus, by transitivity, each such polynomials are congruent to each other. Further, by adding a constant term, we have

$$(\sum_{m=0}^{\infty} r_m X^m, r_0) \in \Omega,$$

which means that each two polynomials with the same constant terms are congruent with each other.

Consider next the same situation from the point of view of Proposition 5.38. It says that the elements of Ω are of the form

$$\text{(5.2)} \quad \Big(\sum_{m,n\in\mathbb{N}} H_{m,n}X^m0^n, \sum_{m,n\in\mathbb{N}} H'_{m,n}X^m0^n\Big) = \Big(\sum_{m\in\mathbb{N}} H_{m,0}X^m, \sum_{m\in\mathbb{N}} H'_{m,0}X^m\Big),$$

where, for all $m, n \in \mathbb{N}$, $H_{m,n}, H'_{m,n} \in R[X]$, all but finitely many of which are zero, and

$$\sum_{n\in\mathbb{N}} H_{0,n}X^n = \sum_{m,n\in\mathbb{N}} H_{m,n}X^n0^m = \sum_{m,n\in\mathbb{N}} H'_{m,n}X^n0^m = \sum_{n\in\mathbb{N}} H'_{0,n}X^n.$$

The polynomials above are the same, exactly when it holds that $H_{0,n} = H'_{0,n}$ for all $n \in \mathbb{N}$. Then especially $H_{0,0} = H'_{0,0}$, and thus, the constant terms of $H_{0,0}$ and $H'_{0,0}$ must be the same. Since these summands are the only ones that may comprise constant terms, also the constant terms of the above polynomials are the same. Consequently, the constant terms in the pairs of polynomials in (5.2) are the same.

Note that the equality between the rest of the coefficients, $H_{0,n}$ and $H'_{0,n}$, for which $n \neq 0$, is not interesting, since the monomials with these coefficients do not exist in the component polynomials of the formal pair in (5.2).

Remark. In the case of several indeterminants, we can prove a similar kind of result. If R is a semiring, then the congruence $\langle (X_1, 0), \ldots, (X_n, 0)\rangle$ on $R[X_1, \ldots, X_n]$ consists of those pairs of polynomials that have the same constant terms.

In the case of a number semiring and of the congruence $\langle (a, 0) \rangle$, where a is an element of the semiring, we end up to a similar kind of equality condition,

$$\sum_{n \in \mathbb{N}} r_{0,n} a^n = \sum_{n \in \mathbb{N}} s_{0,n} a^n,$$

but now we cannot conclude the coefficients to be pairwise equal. Especially in the case of max-plus algebra, only the maximal summands are required to be the same.

Indeterminants are special kind of elements in a polynomial semiring, which makes the congruences of the form $\langle (X, 0) \rangle$ interesting. Instead, other congruences of the form $\langle (a, 0) \rangle$, where a is a usual element of a semiring, are not very practical, as will be shown in the following lemma.

Lemma 5.41. *Let R be a tropical semiring (with max-plus algebra). If $a \geq 1_R$, then $\langle (a, 0_R) \rangle = R \times R$.*

Proof. We provide two different proofs for the claim. One is based on Proposition 5.38, while the other utilizes the properties of congruences.

Denote $0 := 0_R$ and $1 := 1_R$, but note that the exponents below are natural numbers. Now,

$$(1, 0) = (1 + 0, 0 + 0) = (1 \cdot a^0 \cdot 0^0 + 1 \cdot a^0 \cdot 0^1, 0 \cdot a^0 \cdot 0^0 + 1 \cdot a^0 \cdot 0^1),$$

and

$$1 \cdot a^0 \cdot 0^0 + 1 \cdot a^1 \cdot 0^0 = 1 + a = a = 0 + a = 0 \cdot a^0 \cdot 0^0 + 1 \cdot a^1 \cdot 0^0,$$

when assuming that $a \geq 1$. Proposition 5.38 implies $(1, 0) \in \langle (a, 0) \rangle$, when the claim follows from Lemma 5.31.

The other proof proceeds as follows. Clearly, $(a, 0), (1, 1) \in \langle (a, 0) \rangle$. By taking the sum between these elements, we obtain $(a, 1) \in \langle (a, 0) \rangle$, when assuming that $a \geq 1$. By symmetry, $(1, a) \in \langle (a, 0) \rangle$, and by applying transitivity to $(1, a)$ and $(a, 0)$, we obtain $(1, 0) \in \langle (a, 0) \rangle$. The claim follows again from Lemma 5.31. $\square$

Remark. The claim does not hold for indeterminants in a polynomial semiring, as shown in Example 5.40. On the other hand, the assumption $X \geq 1$ does not hold, either.

If R is a semifield, the claim holds true even without the assumption $a \geq 1$. Namely, if $0 \neq a \in R$, then $(a, 0), (a^{-1}, a^{-1}) \in \langle (a, 0) \rangle$, and by taking the product between these elements, we obtain $(1, 0) \in \langle (a, 0) \rangle$.

5.6 Congruences related to ideals

Recall the equality between cosets, as discussed at the beginning of Section 5.3 and in Example 5.18. Such a relation is a congruence, and thus, it provides a connection between congruences and ideals. However, the connection holds true with rings, but not with semirings, and thus, we turn temporarily to rings. The proof of the connection utilizes the following lemma.

Lemma 5.42. *Let R be a ring and $I \subset R$ an ideal. Then*

$$\langle (f,0) \rangle_{f \in I} = \{(f_i, g_i) \in R \times R \mid f_i - g_i \in I\}.$$

Proof. Denote the set on the right as Ω, and suppose that $(f_i, g_i) \in \Omega$. Then $f_i - g_i \in I$, and thus, $(f_i - g_i, 0) \in \langle (f,0) \rangle_{f \in I}$ (as a generator). Clearly, $(g_i, g_i) \in \langle (f,0) \rangle_{f \in I}$, and when taking the sum between these elements, we obtain

$$(f_i, g_i) = ((f_i - g_i) + g_i, 0 + g_i) \in \langle (f,0) \rangle_{f \in I}.$$

To show the other direction, note that Ω can be written in an equivalent form as

$$\Omega = \{(f_i, g_i) \in R \times R \mid f_i + I = g_i + I\}.$$

It is easy to see that Ω is a congruence (as told in Example 5.18). It is also clear that $(f,0) \in \Omega$ for all $f \in I$. Since $\langle (f,0) \rangle_{f \in I}$ is the smallest congruence containing the elements of the form $(f,0)$, where $f \in I$, it holds that $\langle (f,0) \rangle_{f \in I} \subset \Omega$. $\square$

Proposition 5.43. *Let R be a ring. There is a bijection between ideals in R and congruences generated by relations of the form $(f,0)$, for $f \in R$. The bijection is given as*

$$\varphi : I \mapsto \langle (f,0) \rangle_{f \in I}.$$

Proof. Suppose that $I \subset R$ is an ideal. Then based on Lemma 5.42,

$$\langle (f,0) \rangle_{f \in I} = \{(f_i, g_i) \in R \times R \mid f_i - g_i \in I\}.$$

Define the map

$$\psi : \{(f_i, g_i) \in R \times R \mid f_i - g_i \in I\} \mapsto \{f_i - g_i \mid f_i - g_i \in I, \text{ for } f_i, g_i \in R\}.$$

Now ψ is the inverse map of φ, since

$$\begin{aligned}
&\varphi(\psi(\{(f_i, g_i) \in R \times R \mid f_i - g_i \in I\})) \\
&= \varphi(\{f_i - g_i \mid f_i - g_i \in I, \text{ for } f_i, g_i \in R\}) \\
&= \varphi(I) \\
&= \langle (f,0) \rangle_{f \in I} \\
&= \{(f_i, g_i) \in R \times R \mid f_i - g_i \in I\},
\end{aligned}$$

and

$$\begin{aligned}\psi(\varphi(I)) &= \psi(\langle (f,0)\rangle_{f\in I}) = \psi(\{(f_i,g_i)\in R\times R \mid f_i - g_i \in I\}) \\ &= \{f_i - g_i \mid f_i - g_i \in I, \text{ for } f_i, g_i \in R\} = I.\end{aligned}$$

□

Remark. Alternatively, the above claim can be proved by using Proposition 5.38. This is done by showing φ to be both injective and surjective.

To show φ to be injective, let I and J be ideals of R, and assume that

$$\langle (f,0)\rangle_{f\in I} = \langle (g,0)\rangle_{g\in J}.$$

The task is to show that $I = J$. Suppose that $f_0 \in I$. Then clearly, $(f_0, 0) \in \langle (f,0)\rangle_{f\in I}$, when by assumption, $(f_0,0) \in \langle (g,0)\rangle_{g\in J}$.

Based on Proposition 5.38, the elements of $\langle (g,0)\rangle_{g\in J}$ are exactly those of the form

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(J)}} h_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{g}^{\boldsymbol{m}}\boldsymbol{0}^{\boldsymbol{n}}, \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(J)}} h'_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{g}^{\boldsymbol{m}}\boldsymbol{0}^{\boldsymbol{n}}\Big) = \Big(\sum_{\boldsymbol{m}\in\mathbb{N}^{(J)}} h_{\boldsymbol{m},\boldsymbol{0}}\boldsymbol{g}^{\boldsymbol{m}}, \sum_{\boldsymbol{m}\in\mathbb{N}^{(J)}} h'_{\boldsymbol{m},\boldsymbol{0}}\boldsymbol{g}^{\boldsymbol{m}}\Big),$$

where, for all $\boldsymbol{m},\boldsymbol{n} \in \mathbb{N}^{(J)}$, $h_{\boldsymbol{m},\boldsymbol{n}}, h'_{\boldsymbol{m},\boldsymbol{n}} \in R$, all but finitely many of which are zero, and

$$\sum_{\boldsymbol{n}\in\mathbb{N}^{(J)}} h_{\boldsymbol{0},\boldsymbol{n}}\boldsymbol{g}^{\boldsymbol{n}} = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(J)}} h_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{g}^{\boldsymbol{n}}\boldsymbol{0}^{\boldsymbol{m}} = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(J)}} h'_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{g}^{\boldsymbol{n}}\boldsymbol{0}^{\boldsymbol{m}} = \sum_{\boldsymbol{n}\in\mathbb{N}^{(J)}} h'_{\boldsymbol{0},\boldsymbol{n}}\boldsymbol{g}^{\boldsymbol{n}}.$$

This equation can be written in an equivalent way as

$$\sum_{\boldsymbol{0}\neq\boldsymbol{n}\in\mathbb{N}^{(J)}} h_{\boldsymbol{0},\boldsymbol{n}}\boldsymbol{g}^{\boldsymbol{n}} + h_{\boldsymbol{0},\boldsymbol{0}} = \sum_{\boldsymbol{0}\neq\boldsymbol{n}\in\mathbb{N}^{(J)}} h'_{\boldsymbol{0},\boldsymbol{n}}\boldsymbol{g}^{\boldsymbol{n}} + h'_{\boldsymbol{0},\boldsymbol{0}}.$$

Since R is a ring with additive inverses, the equation can be further written as

$$h_{\boldsymbol{0},\boldsymbol{0}} - h'_{\boldsymbol{0},\boldsymbol{0}} = \sum_{\boldsymbol{0}\neq\boldsymbol{n}\in\mathbb{N}^{(J)}} (h'_{\boldsymbol{0},\boldsymbol{n}} - h_{\boldsymbol{0},\boldsymbol{n}})\boldsymbol{g}^{\boldsymbol{n}}.$$

Each summand on the right is a product with some $g \in J$ as its multiplier. Therefore the sum is a linear combination over the elements of J, and thus, $h_{\boldsymbol{0},\boldsymbol{0}} - h'_{\boldsymbol{0},\boldsymbol{0}} \in J$.

Consider next the form of the elements of $\langle (g,0)\rangle_{g\in J}$, as written in the beginning of the proof based on Proposition 5.38. Due to the existence of additive inverses in R, we can proceed with the difference between the components of the pair, i.e.

$$\begin{aligned}&\sum_{\boldsymbol{m}\in\mathbb{N}^{(J)}} h_{\boldsymbol{m},\boldsymbol{0}}\boldsymbol{g}^{\boldsymbol{m}} - \sum_{\boldsymbol{m}\in\mathbb{N}^{(J)}} h'_{\boldsymbol{m},\boldsymbol{0}}\boldsymbol{g}^{\boldsymbol{m}} \\ &= \sum_{\boldsymbol{0}\neq\boldsymbol{m}\in\mathbb{N}^{(J)}} h_{\boldsymbol{m},\boldsymbol{0}}\boldsymbol{g}^{\boldsymbol{m}} + h_{\boldsymbol{0},\boldsymbol{0}} - \Big(\sum_{\boldsymbol{0}\neq\boldsymbol{m}\in\mathbb{N}^{(J)}} h'_{\boldsymbol{m},\boldsymbol{0}}\boldsymbol{g}^{\boldsymbol{m}} + h'_{\boldsymbol{0},\boldsymbol{0}}\Big) \\ &= \underbrace{\sum_{\boldsymbol{0}\neq\boldsymbol{m}\in\mathbb{N}^{(J)}} h_{\boldsymbol{m},\boldsymbol{0}}\boldsymbol{g}^{\boldsymbol{m}}}_{\in J} - \underbrace{\sum_{\boldsymbol{0}\neq\boldsymbol{m}\in\mathbb{N}^{(J)}} h'_{\boldsymbol{m},\boldsymbol{0}}\boldsymbol{g}^{\boldsymbol{m}}}_{\in J} + \underbrace{h_{\boldsymbol{0},\boldsymbol{0}} - h'_{\boldsymbol{0},\boldsymbol{0}}}_{\in J}.\end{aligned}$$

Both the above sums are again linear combinations over elements of J, and thus, they are elements of J themselves. Therefore the subtraction between the components of the pairs in $\langle(g,0)\rangle_{g\in J}$ is an element of J. Since $(f_0,0)\in\langle(g,0)\rangle_{g\in J}$, we have

$$f_0 = f_0 - 0 \in J.$$

We have now shown that $I\subset J$, and the other direction can be proved in a similar way.

We will finally prove φ to be surjective. Let Ω be a congruence generated by relations of the form $(f,0)$, where $f\in I$, for $I\subset R$ a subset. In other words, $\Omega=\langle(f,0)\rangle_{f\in I}$. Now I is not necessarily an ideal, but we can take an ideal generated by I, when it holds $\Omega=\varphi(\langle I\rangle)$. Namely,

$$\langle(f,0)\rangle_{f\in I} = \langle(f,0)\rangle_{f\in\langle I\rangle}, \tag{5.3}$$

which can be proved as follows. Suppose that $g\in\langle I\rangle\backslash I$. We will show that $(g,0)\in\langle(f,0)\rangle_{f\in I}$. Since $g\in\langle I\rangle$, it can be expressed as $g=\sum_{f\in I}a_f f$, where $a_f\in R$, for all $f\in I$. Now

$$(g,0) = (\sum_{f\in I}a_f f, 0) = \sum_{f\in I}(a_f,a_f)(f,0)\in\langle(f,0)\rangle_{f\in I}.$$

This proves the sets in (5.3) to be the same, which further proves φ to be surjective.

Example 5.44. Consider $\mathbb{Z}$ as a ring and its principal ideals $\langle n\rangle$, for $n\in\mathbb{Z}$. Each ideal $\langle n\rangle$ corresponds bijectively to a congruence $\langle(n,0)\rangle$. Namely, the elements of $\langle n\rangle$ are of the form kn, where $k\in\mathbb{Z}$. Each $(kn,0)\in\langle(n,0)\rangle$, since $(n,0)$ and (k,k) do. Therefore, a congruence corresponding to a principal ideal is generated by a single element, and we can write the bijection proved in Proposition 5.43 in the form

$$\varphi:\langle n\rangle\mapsto\langle(n,0)\rangle.$$

Remark. More generally, if R is a semiring and $a\in R$, then

$$\langle(a,0)\rangle_{a\in\langle a\rangle} = \langle(a,0)\rangle,$$

which can be proved in the same way as above.

Example 5.45. Proposition 5.43 holds for rings, but not for semirings. To see this, consider the semiring $R:=(\mathbb{N}\cup\{-\infty\},\oplus,\odot)$ and its principal ideals $\langle n\rangle$, for $n\in R$. Now

$$\langle n\rangle = \{k\odot n \mid k\in R\} = \{k+n\mid k\in R\} = \{l\mid l\geq n, l\in R\}\cup\{-\infty\}.$$

Therefore if $m,n\in\mathbb{N}$ such that $m>n$, then $\langle m\rangle\subsetneq\langle n\rangle$, but based on Lemma 5.41, $\langle(m,0)\rangle=\langle(n,0)\rangle$.

Chapter 6

Tropical congruence varieties

6.1 Congruence varieties

This chapter exploits the concepts introduced in the previous chapter in order to show the correspondence between algebraic sets and congruence varieties. Recall Definition 5.17 for a congruence, and Corollary 5.21 for a more practical description for it. Note also that Definition 4.10 actually gives a congruence. We will next define a similar kind of congruence, but now we are interested in polynomials and polynomial functions that give the same values on a certain subset (e.g. an interval) of a domain, without (most often) paying attention to the essentiality of the monomials.

The congruence to be defined is based on ν-equivalence relation, given in Definition 3.23. According to Proposition 3.26, it respects addition and multiplication in a correct way, and thus, it is a congruence. This was already considered in Example 5.19.

Most often in this chapter, we will use a 1-semifield as an algebraic structure to enable geometric interpretation. We reject 1-semifields† here, since this chapter is strongly based on congruences that were defined for a semiring (with zero), although such a restriction would not have always been necessary. Moreover, we will apply twisted product, when the existence of zero is more natural.

We give below definitions for a congruence variety separately based on (polynomial) functions and on polynomials.

Definition 6.1. Let K be a 1-semifield, $S \subset K^n$ a set and $X \subset S$ a subset. The set

$$\Omega_X := \{(f, g) \in \mathrm{Fun}(S, K) \times \mathrm{Fun}(S, K) \mid f(a) \cong_\nu g(a), \text{ for all } a \in X\}$$

is called a *congruence of* X *on* $\mathrm{Fun}(S, K)$.

Conversely, let Ω be a relation on $\mathrm{Fun}(S, K)$, and $f, g \in \mathrm{Fun}(S, K)$. Define the set

$$V(\Omega) := \{a \in S \mid f(a) \cong_\nu g(a), \text{ for all } (f, g) \in \Omega\}.$$

If Ω is a congruence, the above set is called the *congruence variety of* Ω.

If Ω is unambiguous or meaningless, we speak on *congruence variety* or a *variety of a congruence.*

Definition 6.2. Let K be a 1-semifield, $S \subset K^n$ a set and $X \subset S$ a subset. The set

$$\Omega_X := \{(F, G) \in (K[X_1, \ldots, X_n])^2 \mid F(a) \cong_\nu G(a), \text{ for all } a \in X\}$$

is called a *congruence of* X *on* $K[X_1, \ldots, X_n]$.

Let Ω be a relation on $K[X_1, \ldots, X_n]$, and $F, G \in K[X_1, \ldots, X_n]$. Define the set

$$V(\Omega) := \{a \in S \mid F(a) \cong_\nu G(a), \text{ for all } (F, G) \in \Omega\}.$$

If Ω is a congruence, the above set is called the *congruence variety of* Ω *(with respect to* S*).*

If Ω is unambiguous or meaningless, we speak on *congruence variety* or a *variety of a congruence.*

Example 6.3. Let K be a 1-semifield, and Ω a congruence on $K[X_1, \ldots, X_n]$. Then $\Omega_\emptyset = K[X_1, \ldots, X_n] \times K[X_1, \ldots, X_n]$.

Moreover, $V(\Omega_\emptyset) = \emptyset$. Namely, $a \in V(\Omega_\emptyset)$ exactly when $F(a) \cong_\nu G(a)$ for all $(F, G) \in \Omega_\emptyset = K[X_1, \ldots, X_n] \times K[X_1, \ldots, X_n]$. There is no element a to satisfy this condition, and thus, $V(\Omega_\emptyset) = \emptyset$.

A congruence can never be the empty set, since reflexive relations are always elements of it. If Ω is this kind of congruence, i.e. a trivial congruence (as introduced in the remark after Proposition 5.26), then $V(\Omega) = K^n$.

On the other hand, if $\Omega \subset K[X_1, \ldots, X_n] \times K[X_1, \ldots, X_n]$, then $V(\Omega) = K^n$, exactly when Ω consists of such pairs of polynomials that are the same or differ from each other only by those monomials that are not essential. In other words, Ω is the congruence given in Definition 4.10.

Example 6.4. Let K be a 1-semifield, $S \subset K^n$ a subset and Ω a congruence of S on $K[X_1, \ldots, X_n]$. If $\Omega = \langle (F, cF) \rangle$, where $F \in K[X_1, \ldots, X_n]$ and $1_K \neq c \in K$, then Proposition 5.38 implies the elements of Ω to be of the form

$$\begin{aligned} &\Big(\sum_{m,n \in \mathbb{N}} H_{m,n} F^m (cF)^n, \sum_{m,n \in \mathbb{N}} H'_{m,n} F^m (cF)^n \Big) \\ =&\Big(\sum_{m,n \in \mathbb{N}} H_{m,n} c^n F^{m+n}, \sum_{m,n \in \mathbb{N}} H'_{m,n} c^n F^{m+n} \Big), \end{aligned}$$

where, for all $m, n \in \mathbb{N}$, $H_{m,n}, H'_{m,n} \in K[X_1, \ldots, X_n]$, all but finitely many of which are zero, and

$$\sum_{m,n \in \mathbb{N}} H_{m,n} c^m F^{m+n} = \sum_{m,n \in \mathbb{N}} H'_{m,n} c^m F^{m+n}.$$

Therefore we have

$$V(\Omega) = \begin{cases} \{0_K\}, & \text{if } F \text{ has no constant term,} \\ \emptyset, & \text{if } F \text{ has a constant term.} \end{cases}$$

To see this, consider the first case first. Even if F has no constant term, Ω contains pairs of polynomials with constant terms. However, if $(G, G') \in \Omega$, for $G, G \in K[X_1, \ldots, X_n]$, then the constant terms of G and G' are the same. Namely, we can write G in the form

$$\begin{aligned} G &= \sum_{m,n \in \mathbb{N}} H_{m,n} c^n F^{m+n} = \sum_{0 \neq m, 0 \neq n \in \mathbb{N}} H_{m,n} c^n F^{m+n} + H_{0,0} c^0 F^{0+0} \\ &= \sum_{0 \neq m, 0 \neq n \in \mathbb{N}} H_{m,n} c^n F^{m+n} + H_{0,0}, \end{aligned}$$

and assume the corresponding representation for G'. Since F has no constant term, the part of the sum, where $m \neq 0$ and $n \neq 0$, cannot have a constant term. Therefore the constant term of G is the same as the constant term of $H_{0,0}$, as well as the constant term of G' is the same as the constant term of $H'_{0,0}$.

The equality condition can be written in the form

$$\sum_{0 \neq m, 0 \neq n \in \mathbb{N}} H_{m,n} c^m F^{m+n} + H_{0,0} = \sum_{0 \neq m, 0 \neq n \in \mathbb{N}} H'_{m,n} c^m F^{m+n} + H'_{0,0},$$

which requires the constant terms of $H_{0,0}$ and $H'_{0,0}$ to be the same, since these summands are again the only parts of sums, which can have a constant term. Therefore the constant terms of G and G' must be the same, as well.

Polynomials with the same constant terms reach ν-equivalent values at 0_K. Moreover, this is the only point, where all the above kind of pairs of polynomials are ν-equivalent. Hence, in this case, $V(\Omega) = \{0_K\}$.

If F instead has a constant term, then it holds either $F(a) <_\nu cF(a)$ for all $a \in S$, or $F(a) >_\nu cF(a)$ for all $a \in S$, when recalling that $c \neq 1_K$. Hence, in this case, $V(\Omega) = \emptyset$.

Based on Proposition 5.22, an intersection of congruences is again a congruence. Especially in the case of congruences of certain sets, we can describe the intersection more precisely. This is done in the following proposition [18, p. 31].

Proposition 6.5. *Let K be a 1-semifield, $X, Y \subset K^n$ subsets, and Ω_X, Ω_Y and $\Omega_{X \cup Y}$ congruences on $K[X_1, \ldots, X_n]$. Then*

$$\Omega_X \cap \Omega_Y = \Omega_{X \cup Y}.$$

Proof. Suppose that $F, G \in K[X_1, \dots, X_n]$. Now,

$$\begin{aligned}
&(F,G) \in \Omega_X \cap \Omega_Y \\
&\iff (F,G) \in \Omega_X \text{ and } (F,G) \in \Omega_Y \\
&\iff F(a) \cong_\nu G(a), \text{ for all } a \in X \quad \text{and} \quad F(a) \cong_\nu G(a), \text{ for all } a \in Y \\
&\iff F(a) \cong_\nu G(a), \text{ for all } a \in X \cup Y \\
&\iff (F,G) \in \Omega_{X \cup Y}.
\end{aligned}$$

□

We will next show that there is a bijection between varieties of congruences and congruences of varieties. We start with the properties familiar from usual algebraic geometry.

Proposition 6.6. *Let K be a 1-semifield, $X, Y \subset K^n$ subsets, and Ω_X and Ω_Y congruences on $K[X_1, \dots, X_n]$. If $X \subset Y$, then $\Omega_Y \subset \Omega_X$. On the other hand, if Ω and Ω' are relations on $K[X_1, \dots, X_n]$ such that $\Omega \subset \Omega'$, then $V(\Omega') \subset V(\Omega)$.*

Proof. Suppose that $(F, G) \in \Omega_Y$, for $F, G \in K[X_1, \dots, X_n]$. Therefore $F(a) \cong_\nu G(a)$, for all $a \in Y$. If $X \subset Y$, then especially $F(a) \cong_\nu G(a)$, for all $a \in X$. Hence, $(F, G) \in \Omega_X$.

Suppose that $a \in V(\Omega')$. Therefore $F(a) \cong_\nu G(a)$, for all $(F, G) \in \Omega'$. If $\Omega \subset \Omega'$, then especially $F(a) \cong_\nu G(a)$, for all $(F, G) \in \Omega$. Hence, $a \in V(\Omega)$. □

Remark. By combining the above two claims, it holds that

$$X \subset Y \quad \text{implies} \quad V(\Omega_X) \subset V(\Omega_Y).$$

Lemma 6.7. *Let K be a 1-semifield. Then*

(i) *$X \subset V(\Omega_X)$, for all subsets $X \subset K^n$,*

(ii) *$\Omega \subset \Omega_{V(\Omega)}$, for all congruences Ω on $K[X_1, \dots, X_n]$,*

(iii) *if X is a congruence variety, then $X = V(\Omega_X)$.*

Proof. (i) Suppose that $a \in X$ and Ω_X is a congruence of X. Therefore, if $(F, G) \in \Omega_X$, then $F(a) \cong_\nu G(a)$. Hence, $a \in V(\Omega_X)$.

(ii) Suppose that $(F, G) \in \Omega$, for $F, G \in K[X_1, \dots, X_n]$. Therefore, if $a \in V(\Omega)$, then $F(a) \cong_\nu G(a)$. Hence, $(F, G) \in \Omega_{V(\Omega)}$.

(iii) If X is a congruence variety, then it holds $X = V(\Omega)$, for some congruence Ω. When applying the inverse subsets of Proposition 6.6 to (ii), we obtain $V(\Omega_{V(\Omega)}) \subset V(\Omega)$. Since $X = V(\Omega)$, this means that $V(\Omega_X) \subset X$. Together with (i), we have $X = V(\Omega_X)$.

□

Remark. Point (iii) can be written as $V(\Omega) = V(\Omega_{V(\Omega)})$. If X is a congruence variety, point (iii) implies $\Omega_X = \Omega_{V(\Omega_X)}$, when just applying Ω for both sides.

Corollary 6.8. *Let K be a 1-semifield. There is a bijection between varieties of congruences and congruences of varieties, given by*

$$X \mapsto \Omega_X \quad \text{and} \quad \Omega \mapsto V(\Omega),$$

where on the left map, $X \subset K^n$ is a variety of a congruence, while on the right map, Ω is a congruence of a variety on $K[X_1, \ldots, X_n]$.

Proof. Consider the compound map

$$X \mapsto \Omega_X \mapsto V(\Omega_X).$$

It is bijective, since X is a variety of a congruence, and thus, the point (iii) of Lemma 6.7 gives $X = V(\Omega_X)$.

On the other hand, since Ω is a congruence of a variety, we can write $\Omega = \Omega_X$, where X is a variety. Now, consider the map

$$\Omega_X \mapsto V(\Omega_X) \mapsto \Omega_{V(\Omega_X)}.$$

It is bijective, since the remark after Lemma 6.7 gives $\Omega_X = \Omega_{V(\Omega_X)}$. □

Remark. However, the compound map

$$\Omega \mapsto V(\Omega) \mapsto \Omega_{V(\Omega)},$$

where Ω is a congruence (but not a congruence of a variety), is only injective, based on point (ii) of Lemma 6.7.

We will next prove that the intersection and finite union of congruence varieties are congruence varieties. To do so, we need the following lemma.

Lemma 6.9. *Let K be a 1-semifield, and Ω a relation on $K[X_1, \ldots, X_r]$. Then $V(\Omega) = V(\langle\Omega\rangle)$.*

Proof. Clearly, $\Omega \subset \langle\Omega\rangle$, and thus, Proposition 6.6 implies $V(\langle\Omega\rangle) \subset V(\Omega)$.

To prove the other direction, suppose that $a \in V(\Omega)$. Therefore $F_i(a) \cong_\nu G_i(a)$ for all $(F_i, G_i) \in \Omega$ and $i \in I$, for I an index set. Based on Proposition 3.26, ν-equivalence is a congruence, and thus, it respects multiplication. Therefore

$$(\prod_{i \in I} F_i^{m_i})(a) \cong_\nu (\prod_{i \in I} G_i^{m_i})(a),$$

where $m_i \in \mathbb{N}$ for all $i \in I$ such that only finitely many of the exponents (m_i) differ from zero. As a congruence, ν-equivalence is symmetric, and thus, we also have

$$(\prod_{i \in I} G_i^{n_i})(a) \cong_\nu (\prod_{i \in I} F_i^{n_i})(a),$$

where $n_i \in \mathbb{N}$ for all $i \in I$ such that only finitely many of the exponents (n_i) differ from zero. Furthermore, by multiplying the previous expressions together side by side, we obtain

$$(\prod_{i\in I} F_i^{m_i} G_i^{n_i})(a) \cong_\nu (\prod_{i\in I} F_i^{n_i} G_i^{m_i})(a).$$

As a congruence, ν-equivalence is reflexive, and thus, $H_{\boldsymbol{m},\boldsymbol{n}}(a) \cong_\nu H_{\boldsymbol{m},\boldsymbol{n}}(a)$, for all $H_{\boldsymbol{m},\boldsymbol{n}} \in K[X_1, \dots, X_r]$, where $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^{(I)}$. Since ν-equivalence also respects addition, we obtain

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H_{\boldsymbol{m},\boldsymbol{n}} \prod_{i\in I} F_i^{m_i} G_i^{n_i}\Big)(a) \cong_\nu \Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H_{\boldsymbol{m},\boldsymbol{n}} \prod_{i\in I} F_i^{n_i} G_i^{m_i}\Big)(a),$$

for $\boldsymbol{m} = (\dots, m_i, \dots), \boldsymbol{n} = (\dots, n_i, \dots) \in \mathbb{N}^{(I)}$, where only finitely many of the exponents (m_i and n_i) differ from zero, and for $H_{\boldsymbol{m},\boldsymbol{n}} \in K[X_1, \dots, X_r]$, all but finitely many of which are zero. By denoting

$$\boldsymbol{F}^{\boldsymbol{m}}\boldsymbol{G}^{\boldsymbol{n}} = \prod_{i\in I} F_i^{m_i} G_i^{n_i} \quad \text{and} \quad \boldsymbol{F}^{\boldsymbol{n}}\boldsymbol{G}^{\boldsymbol{m}} = \prod_{i\in I} F_i^{n_i} G_i^{m_i},$$

the previous form can be written in a shorter way, as

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{m}}\boldsymbol{G}^{\boldsymbol{n}}\Big)(a) \cong_\nu \Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{n}}\boldsymbol{G}^{\boldsymbol{m}}\Big)(a).$$

The remaining task is to consider the transitive relations occuring in $\langle\Omega\rangle$, although possibly missing in Ω. Until now, we have concluded that the above ν-equivalence holds for all $H_{\boldsymbol{m},\boldsymbol{n}} \in K[X_1, \dots, X_r]$. Therefore

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{m}}\boldsymbol{G}^{\boldsymbol{n}}\Big)(a) \cong_\nu \Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{n}}\boldsymbol{G}^{\boldsymbol{m}}\Big)(a),$$

and

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H'_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{m}}\boldsymbol{G}^{\boldsymbol{n}}\Big)(a) \cong_\nu \Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H'_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{n}}\boldsymbol{G}^{\boldsymbol{m}}\Big)(a),$$

where for all $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^{(I)}$, $H_{\boldsymbol{m},\boldsymbol{n}}, H'_{\boldsymbol{m},\boldsymbol{n}} \in K[X_1, \dots, X_r]$, all but finitely many of which differ from zero. If

$$\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{n}}\boldsymbol{G}^{\boldsymbol{m}} = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H'_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{n}}\boldsymbol{G}^{\boldsymbol{m}}, \tag{6.1}$$

then based on symmetry and transitivity of ν-equivalence, it also holds

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{m}}\boldsymbol{G}^{\boldsymbol{n}}\Big)(a) \cong_\nu \Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H'_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{m}}\boldsymbol{G}^{\boldsymbol{n}}\Big)(a).$$

On the other hand, based on Proposition 5.38, the elements of the form

$$\Big(\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{m}}\boldsymbol{G}^{\boldsymbol{n}}, \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^{(I)}} H'_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{F}^{\boldsymbol{m}}\boldsymbol{G}^{\boldsymbol{n}}\Big),$$

on condition (6.1) are exactly the elements of $\langle\Omega\rangle$. Therefore, we have now proved that $F(a) \cong_\nu G(a)$ holds for all $(F, G) \in \langle\Omega\rangle$. Hence $a \in V(\langle\Omega\rangle)$. $\square$

Lemma 6.10. *Let K be a 1-semifield, and $(\Omega_i)_{i\in I}$ a family of congruences on $K[X_1,\ldots,X_n]$. An intersection of congruence varieties is again a congruence variety, and*

$$\bigcap_{i=1}^{\infty} V(\Omega_i) = V(\bigcup_{i=1}^{\infty}\Omega_i).$$

Especially if each $\Omega_i = \Omega_{X_i}$, for some congruence variety $X_i \subset K^n$, then

$$\bigcap_{i=1}^{\infty} V(\Omega_{X_i}) = V(\Omega_{\bigcap_{i=1}^{\infty} X_i}).$$

Proof. We will first concentrate on the first equation in the claim. Clearly, $\Omega_i \subset \cup_{i=1}^{\infty}\Omega_i$, for all $i \in I$, and thus, based on Proposition 6.6, $V(\cup_{i=1}^{\infty}\Omega_i) \subset V(\Omega_i)$, for all $i \in I$. Therefore $V(\cup_{i=1}^{\infty}\Omega_i) \subset \cap_{i=1}^{\infty}V(\Omega_i)$.

To prove the other direction, suppose that $a \in \cap_{i=1}^{\infty}V(\Omega_i)$. Therefore $a \in V(\Omega_i)$, for all $i \in I$. This means that $F(a) \cong_\nu G(a)$ for all $(F,G) \in \Omega_i$ for all $i \in I$, and thus, $F(a) \cong_\nu G(a)$ for all $(F,G) \in \cup_{i=1}^{\infty}\Omega_i$. Hence $a \in V(\cup_{i=1}^{\infty}\Omega_i)$.

Since based on Lemma 6.9, $V(\bigcup_{i=1}^{\infty}\Omega_i) = V(\langle\bigcup_{i=1}^{\infty}\Omega_i\rangle)$, we have now, besides the first equation, proved also the intersection of congruence varieties to be a congruence variety.

We will finally concentrate on the second equation. Since now each X_i is a congruence variety, we have $X_i = V(\Omega_{X_i})$ for all $i \in I$. Moreover, based on the first part of the current proof, an intersection of congruence varieties is a congruence variety. Therefore

$$\bigcap_{i=1}^{\infty} V(\Omega_{X_i}) = \bigcap_{i=1}^{\infty} X_i = V(\Omega_{\bigcap_{i=1}^{\infty} X_i}).$$

□

Remark. By combining the above claims, we obtain

$$V(\bigcup_{i=1}^{\infty}\Omega_{X_i}) = \bigcap_{i=1}^{\infty} V(\Omega_{X_i}) = V(\Omega_{\bigcap_{i=1}^{\infty} X_i}),$$

for congruence varieties X_i.

There is another way to express intersections of congruence varieties, assuming that the zero element is included in the semiring in question.

Lemma 6.11. *Let K be a 1-semifield (i.e. $0 \in K$), and $(\Omega_i)_{i\in I}$ a family of congruences on $K[X_1,\ldots,X_n]$. Then*

$$\bigcap_{i=1}^{\infty} V(\Omega_i) = V(\sum_{i=1}^{\infty}\Omega_i).$$

Proof. Since $0 \in K$, it holds that $\Omega_i \subset \sum_{i=1}^{\infty} \Omega_i$, for all $i \in I$. Namely, each $(F, G) \in \Omega_i$ can be written as $(F + 0 + \dots, G + 0 + \dots)$, which is an element of $\sum_{i=1}^{\infty} \Omega_i$. Therefore, we clearly have

$$\bigcup_{i=1}^{\infty} \Omega_i \subset \sum_{i=1}^{\infty} \Omega_i \subset \langle \bigcup_{i=1}^{\infty} \Omega_i \rangle,$$

and when applying Proposition 6.6, we obtain

$$V(\langle \bigcup_{i=1}^{\infty} \Omega_i \rangle) \subset V(\sum_{i=1}^{\infty} \Omega_i) \subset V(\bigcup_{i=1}^{\infty} \Omega_i).$$

Based on Lemma 6.9, the first and the last set are the same, and thus, all the above sets are the same. Finally, the claim follows from Lemma 6.10. □

Remark. As a combination of Lemmata 6.10 and 6.11, we obtain

$$V(\bigcup_{i=1}^{\infty} \Omega_i) = \bigcap_{i=1}^{\infty} V(\Omega_i) = V(\sum_{i=1}^{\infty} \Omega_i).$$

Recall the twisted product of congruences from Definition 5.32 to be applied in the following lemma. The proof of the lemma follows that of Lemma 5.36. However, since we below have ν-relations and consider polynomials on a certain value, we write the same steps again instead of referring to the earlier lemma. These steps follow also those presented in [1, p. 10] (Lemma 3.2).

Lemma 6.12. *Let K be a 1-semifield, and $\Omega, \Omega' \subset (K[X_1, \dots, X_n])^2$ congruences. A finite union of congruence varieties is again a congruence variety, and*

$$V(\Omega) \cup V(\Omega') = V(\Omega \star \Omega').$$

Proof. We will first concentrate on the equation in the claim. Based on Lemma 5.35, we have both $\Omega \star \Omega' \subset \Omega$ and $\Omega \star \Omega' \subset \Omega'$. Therefore based on Proposition 6.6, we obtain both $V(\Omega) \subset V(\Omega \star \Omega')$ and $V(\Omega') \subset V(\Omega \star \Omega')$, and thus, $V(\Omega) \cup V(\Omega') \subset V(\Omega \star \Omega')$.

To show the other direction, suppose that $a \in V(\Omega \star \Omega')$. Therefore

$$(FF' + GG')(a) \cong_\nu (FG' + GF')(a),$$

for all $(FF' + GG', FG' + GF') \in \Omega \star \Omega'$, where $(F, G) \in \Omega$ and $(F', G') \in \Omega'$ such that $F, F', G, G' \in K[X_1, \dots, X_n]$.

Suppose against the claim that $a \notin V(\Omega)$ and $a \notin V(\Omega')$. This means that $F(a) \ncong_\nu G(a)$, for some $(F, G) \in \Omega$ and $F'(a) \ncong_\nu G'(a)$, for some $(F', G') \in \Omega'$. Based on Proposition 3.27, K is totally ordered, and thus, we can assume, for example, that

$$F(a) <_\nu G(a) \quad \text{and} \quad F'(a) <_\nu G'(a),$$

which imply $G(a) >_\nu 0_K$ and $G'(a) >_\nu 0_K$, respectively, when recalling that 0_K is the smallest element in a totally ordered semiring, as discussed in the remark after Example 3.13. As a 1-semifield, K is cancellative, and thus, Proposition 2.26 implies that multiplication with a non-zero element preserves the strict order. This gives

$$(FG')(a) <_\nu (GG')(a) \quad \text{and} \quad (GF')(a) <_\nu (GG')(a),$$

and furthermore $(FG')(a) + (GF')(a) <_\nu (GG')(a)$. By taking into account the ν-equivalence at the beginning, this is the same as

$$(FF')(a) + (GG')(a) <_\nu (GG')(a),$$

which is a contradiction. Hence, it must be $a \in V(\Omega)$ or $a \in V(\Omega')$.

Since $\Omega \star \Omega'$ is a congruence, we have now proved the union of congruence varieties to be a congruence variety. □

Remark. Although we apply twisted product in the above proof, the claim holds true also for a 1-semifield†. In this case, we can trivially assume that $G(a)$ and $G'(a)$ are non-zero.

There is another expression for a finite union of congruence varieties.

Lemma 6.13. *Let K be a 1-semifield, and $\Omega, \Omega' \subset (K[X_1, \ldots, X_n])^2$ congruences. Then*

$$V(\Omega) \cup V(\Omega') = V(\Omega \cap \Omega').$$

Especially if $\Omega = \Omega_X$ and $\Omega' = \Omega_Y$, for some congruence varieties $X, Y \subset K^n$, then

$$V(\Omega_X) \cup V(\Omega_Y) = V(\Omega_{X \cup Y}).$$

Proof. We will first prove the first claim. Based on Lemma 5.35, we have both $\Omega \star \Omega' \subset \Omega$ and $\Omega \star \Omega' \subset \Omega'$, and thus,

$$\Omega \star \Omega' \subset \Omega \cap \Omega' \subset \Omega \quad \text{and} \quad \Omega \star \Omega' \subset \Omega \cap \Omega' \subset \Omega'.$$

Based on Proposition 6.6, these imply

$$V(\Omega) \subset V(\Omega \cap \Omega') \subset V(\Omega \star \Omega') \quad \text{and} \quad V(\Omega') \subset V(\Omega \cap \Omega') \subset V(\Omega \star \Omega').$$

Therefore

$$V(\Omega) \cup V(\Omega') \subset V(\Omega \cap \Omega') \subset V(\Omega \star \Omega').$$

Based on Lemma 6.12, the first and the last sets are the same, and thus, all the above sets are the same.

The second claim follows from

$$V(\Omega_X) \cup V(\Omega_Y) = V(\Omega_X \cap \Omega_Y) = V(\Omega_{X \cup Y}),$$

when applying Proposition 6.5. □

Remark. As a combination of Lemmata 6.12 and 6.13, we obtain

$$V(\Omega \star \Omega') = V(\Omega) \cup V(\Omega') = V(\Omega \cap \Omega').$$

After some examples of congruence varieties (to be presented in the next section), we will show that an infinite union of congruence varieties is not necessarily a conguence variety (see Example 6.20). We end this section with an example of a finite union of congruence varieties.

Example 6.14. Let K be a 1-semifield such that $K_1 = \mathbb{T}$. Suppose that $F \in K[X_1, \dots, X_n]$, and consider the congruences

$$\Omega_2 := \langle (F, 2F) \rangle, \quad \Omega_3 := \langle (F, 3F) \rangle, \quad \Omega_5 := \langle (F, 5F) \rangle, \quad \Omega_7 := \langle (F, 7F) \rangle$$

on $K[X_1, \dots, X_n]$. Now, for each $i \in \{2, 3, 5, 7\}$,

$$V(\Omega_i) = \begin{cases} \{0_K\}, & \text{if } F \text{ has no constant term,} \\ \emptyset, & \text{if } F \text{ has a constant term.} \end{cases}$$

This can be concluded in the same way as in Example 6.4. Therefore the union over congruence varieties $V(\Omega_i)$ is either the empty set or consists only of zero.

Consider next the intersection $\Omega_2 \cap \Omega_3 \cap \Omega_5 \cap \Omega_7$. For example, $(15F, 17F)$ is an element of the intersection. First of all, $(F, 2F), (15, 15) \in \Omega_2$, and thus, the product of them, $(15F, 17F) \in \Omega_2$. Second, $(F, 3F), (14, 14) \in \Omega_3$, and thus, the product of them, $(14F, 17F) \in \Omega_3$. In addition $(15F, 15F) \in \Omega_3$, and thus, the sum $(14F, 17F) \oplus (15F, 15F) = (15F, 17F) \in \Omega_3$. Third, $(F, 5F), (12, 12), (15F, 15F) \in \Omega_5$, and thus, $(F, 5F) \odot (12, 12) \oplus (15F, 15F) = (15F, 17F) \in \Omega_5$. Finally, in a similar way, $(15F, 17F) \in \Omega_7$. Therefore, $V(\Omega_2 \cap \Omega_3 \cap \Omega_5 \cap \Omega_7)$ is either the empty set or consists of only of zero, on the same condition as $V(\Omega_2) \cup V(\Omega_3) \cup V(\Omega_5) \cup V(\Omega_7)$.

6.2 Examples of congruence varieties

6.2.1 Layered point

This section gives some examples of congruence varieties. Since congruence varieties are defined based on ν-equivalence, they have the property that their elements are actually equivalence classes (modulo $\cong_\nu$). In other words, if a is an element of a congruence variety, all the elements that are ν-equivalent to a are also elements of the congruence variety. For this reason, we need the following definition (given, more generally, for a semiring†).

Definition 6.15. Let R be a semiring† and $S \subset R^n$ a subset. A *layered set of S* is the set

$$S^{(\nu)} := \{a \in R^n \mid a \cong_\nu b, \text{ for } b \in S\}.$$

If $S = \{a\}$, we write $\{a^{(\nu)}\}$ rather than $\{a\}^{(\nu)}$. Moreover, $a^{(\nu)}$ stands for any point that is ν-equivalent to a.

If S is a point, interval, line, etc., we call $S^{(\nu)}$, respectively, a *layered point*, *layered interval*, *layered line*, etc.

Remark. If R is not uniform, a layered set can be rather strange: a layered point is actually depicted in Example 3.28. Therefore, geometric considerations can be better understood in the case of a uniform semiring† (1-semifield†).

Lemma 6.16. *A layered point is a congruence variety.*

Proof. Let K be a 1-semifield and $a = (a_1, \ldots, a_n) \in K^n$. Consider polynomials (or rather monomials) over K given as

$$F_1 = X_1, \ldots, F_n = X_n, \quad G_1 = a_1, \ldots, G_n = a_n.$$

Clearly, $F_i(a) = G_i(a)$ for all $i \in \{1, \ldots, n\}$. Based on Proposition 4.13, $F_i(a) \cong_\nu F_i(b)$ and $G_i(a) \cong_\nu G_i(b)$, for all such $b \in K^n$ that satisfy $b \cong_\nu a$. Therefore $F_i(b) \cong_\nu G_i(b)$ for all $b \cong_\nu a$. We claim that

$$\Omega_{\{a^{(\nu)}\}} \supset \{(F_1, G_1), \ldots, (F_n, G_n)\},$$

and furthermore

$$V(\Omega_{\{a^{(\nu)}\}}) = \{a^{(\nu)}\}.$$

Namely, $\Omega_{\{a^{(\nu)}\}}$ consists of all pairs of polynomials that have ν-equivalent values at the points $a^{(\nu)}$. This holds true for (F_1, G_1), since F_1 and G_1 agree on all points, the first coordinate of which is ν-equivalent to a_1. Similarly, F_2 and G_2 agree on points, the second coordinate of which is ν-equivalent to a_2, and so on. Therefore $(F_i, G_i) \in \Omega_{\{a^{(\nu)}\}}$ for all $i \in \{1, \ldots, n\}$.

When constructing $V(\Omega_{\{a^{(\nu)}\}})$, we search for the points, where all pairs of polynomials that are elements of $\Omega_{\{a^{(\nu)}\}}$ reach ν-equivalent values. Especially when considering the pairs (F_i, G_i) and the common values of such pairs, we end up exactly to those points that are ν-equivalent to a.

Hence, $\{a^{(\nu)}\}$ is a congruence variety. □

Example 6.17. Let K be a 1-semifield such that $K_1 = \mathbb{T}$, and $(a, b, c) \in K^3$ a point. Consider six monomials, elements of $K[X, Y, Z]$, given as

$$F_1 = X, \quad F_2 = Y, \quad F_3 = Z, \quad G_1 = a, \quad G_2 = b, \quad G_3 = c.$$

Now,

$$\begin{aligned} F_1(a, y, z) = a = G_1(a, y, z) &\quad \text{for all } y, z \in K, \\ F_2(x, b, z) = b = G_2(x, b, z) &\quad \text{for all } x, z \in K, \\ F_3(x, y, c) = c = G_3(x, y, c) &\quad \text{for all } x, y \in K. \end{aligned}$$

When replacing a, b and c with their ν-equivalent counterparts, the above equations do not necessarily hold, but based on Proposition 4.13 the corresponding ν-equivalences hold. Therefore,

$$(F_1, G_1), (F_2, G_2), (F_3, G_3) \in \Omega_{\{(a,b,c)^{(\nu)}\}}.$$

When searching for such points (x, y, z) that satisfy

$$\begin{aligned} F_1(x, y, z) &\cong_\nu G_1(x, y, z) \quad \text{and} \\ F_2(x, y, z) &\cong_\nu G_2(x, y, z) \quad \text{and} \\ F_3(x, y, z) &\cong_\nu G_3(x, y, z), \end{aligned}$$

we find exactly the points that are ν-equivalent to (a, b, c). Therefore,

$$V(\Omega_{\{(a,b,c)^{(\nu)}\}}) = \{(a, b, c)^{(\nu)}\}.$$

6.2.2 Layered interval

An interval can be defined for a totally ordered set. Based on Proposition 3.27, a 1-semifield is totally ordered. Therefore, we can give the following lemma for any 1-semifield.

Lemma 6.18. *A closed layered interval is a congruence variety.*

Proof. Let K be a 1-semifield, and $A := [a, \dots, b] \subset K_1$, where $a <_\nu b$. Consider the following two polynomials:

$$F = X + a \quad \text{and} \quad G = X + b^{-1}X^2,$$

both of which are elements of $K[X]$. Note that $b^{-1} \in K$, since K is a 1-semifield and $b \in K_1$. We claim that F and G reach ν-equivalent values exactly at the points of $A^{(\nu)}$. The proof for this claim is as follows.

Let $d \in K$. Suppose first that $a \le_\nu d \le_\nu b$. Now,

$$\begin{aligned} F(d^{(\nu)}) &\cong_\nu F(d) = d + a \cong_\nu d, \\ G(d^{(\nu)}) &\cong_\nu G(d) = d + b^{-1}d^2 = b^{-1}d(b + d) \cong_\nu b^{-1}db = d, \end{aligned}$$

where the first ν-equivalences in both lines follow from Proposition 4.13, and the latter ones in both lines from the assumed inequations. Therefore F and G join at all points of $A^{(\nu)}$, as desired. This proves that $(F, G) \in \Omega_{A^{(\nu)}}$.

Suppose next that $d <_\nu a <_\nu b$. By the corresponding reasoning as in the former case, we obtain

$$\begin{aligned} F(d^{(\nu)}) &\cong_\nu F(d) = d + a \cong_\nu a, \\ G(d^{(\nu)}) &\cong_\nu G(d) = d + b^{-1}d^2 = b^{-1}d(b + d) \cong_\nu b^{-1}db = d. \end{aligned}$$

Since $d <_\nu a$, we have $F(d^{(\nu)}) \not\cong_\nu G(d^{(\nu)})$.

Suppose finally that $a <_\nu b <_\nu d$. Again, by the corresponding reasoning as earlier, we have

$$\begin{aligned} F(d^{(\nu)}) &\cong_\nu F(d) = d + a \cong_\nu d, \\ G(d^{(\nu)}) &\cong_\nu G(d) = d + b^{-1}d^2 = b^{-1}d(b+d) \cong_\nu b^{-1}d^2. \end{aligned}$$

Also in this case, $F(d^{(\nu)}) \not\cong_\nu G(d^{(\nu)})$. Namely, if it was $d \cong_\nu b^{-1}d^2$, then multiplying each side with b would give $bd \cong_\nu d^2$. Since K as a 1-semifield is cancellative, we would obtain $b \cong_\nu d$, a contradiction.

Therefore $V(\Omega_{A^{(\nu)}}) = A^{(\nu)}$, since based on the above proof, especially F and G have no other ν-equivalent points than the points of $A^{(\nu)}$. Hence, $A^{(\nu)}$ is a congruence variety. $\square$

Remark. Open intervals or half-open intervals with finite endpoints are not congruence varieties. Namely, suppose that $A =]a, \dots, b[$. Concluded in the same way as in the previous lemma, we obtain $V(\Omega_{A^{(\nu)}}) = [a, \dots, b]^{(\nu)}$, and thus, $A^{(\nu)}$ is not a congruence variety.

Example 6.19. Let K be a 1-semifield such that $K_1 = \mathbb{T}$, and consider the interval $A := [1, \dots, 2] \subset K_1 = \mathbb{T}$. For example, if

$$\begin{aligned} F &= 1 \oplus (-1)X^2 \oplus (-3)X^3 \in K[X], \\ G &= X \oplus (-1)X^2 \oplus (-5)X^4 \in K[X] \end{aligned}$$

(with tangible coefficients), then $(F, G) \in \Omega_{A^{(\nu)}}$, and furthermore, $V(\Omega_{A^{(\nu)}}) = A^{(\nu)}$, since especially F and G have no other ν-equivalent points besides those of $A^{(\nu)}$.

Remark. In the above example, the middle monomials in the polynomials are the same. By setting the monomials with the highest degree or with the lowest degree to be the same, we can see that half-open intervals of the form $]-\infty, \dots, a]$ and $[a, \dots, \infty[$ are congruence varieties.

The next example proceeds with closed intervals and shows that an infinite union of congruence varieties is not necessarily a congruence variety.

Example 6.20. Let L be a sorting semiring† and $K := \mathcal{R}(L, \mathbb{T}^*)$, a uniform (layered) 1-semifield. As a subset of K, consider the infinite union of closed layered intervals $\cup_{i=1}^{\infty}[\frac{1}{i}, 1]$. Then

$$\bigcup_{i=1}^{\infty}[\frac{1}{i}, 1] =]0, 1],$$

where 0 and 1 are real zero and unit, and $\frac{1}{i}$ is a rational number, i.e. the division is a real division, not a tropical one (which would mean a subtraction).

The above equation holds true, since if $a \cong_\nu 0$, then $a \notin [\frac{1}{i}, 1]$ for all $i \in \mathbb{N}^*$. Therefore the above union is not closed. On the other hand, each component of the union, i.e. each closed layered interval is a congruence variety, as proved in Lemma 6.18. However, a half-open layered interval with finite endpoints is not a congruence variety, as discussed in the remark after the same lemma.

6.2.3 Layered line

The following lemma will be needed when proving a layered line to be a congruence variety. It is also related to the discussion in the remark after Example 2.23. Here we will temporarily allow also negative exponents.

Lemma 6.21. *Let K be a 1-semifield, $F, G \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, and $S \subset K^n$ a subset. If $F(a) \cong_\nu G(a)$ holds true for all $a \in S$, then there exist $F', G' \in K[X_1, \ldots, X_n]$ such that also $F'(a) \cong_\nu G'(a)$ holds true for all $a \in S$. Moreover, if F and G are ν-equivalent exactly at the points of S, then the same holds true for F' and G', too.*

Proof. We will first concentrate on the first claim. Write

$$F = \sum_{\boldsymbol{i} \in \mathbb{Z}^n} c_{\boldsymbol{i}} X_1^{i_1} \cdots X_n^{i_n} \quad \text{and} \quad G = \sum_{\boldsymbol{j} \in \mathbb{Z}^n} d_{\boldsymbol{j}} X_1^{j_1} \cdots X_n^{j_n},$$

for $\boldsymbol{i} = (i_1, \ldots, i_n)$ and $\boldsymbol{j} = (j_1, \ldots, j_n)$, and $c_{\boldsymbol{i}}, d_{\boldsymbol{j}} \in K$.

Let $H := X_1^{k_1} \cdots X_n^{k_n} \in K[X_1, \ldots, X_n]$ (when $k_1, \ldots, k_n \in \mathbb{N}$), and define $F' := FH$ and $G' := GH$. Therefore

$$F' = \sum_{\boldsymbol{i} \in \mathbb{Z}^n} c_{\boldsymbol{i}} X_1^{i_1+k_1} \cdots X_n^{i_n+k_n} \quad \text{and} \quad G' = \sum_{\boldsymbol{j} \in \mathbb{Z}^n} d_{\boldsymbol{j}} X_1^{j_1+k_1} \cdots X_n^{j_n+k_n}.$$

In the words of congruences, we have $(F, G) \in \Omega_S$ as well as $(H, H) \in \Omega_S$, and thus, $(F', G') = (FH, GH) \in \Omega_S$. This holds true for any H, i.e. for any exponents $k_1, \ldots, k_n \in \mathbb{N}$. Consequently, we can choose $k_1, \ldots, k_n$ great enough to ensure that $F', G' \in K[X_1, \ldots, X_n]$. Therefore

$$F' = \sum_{\boldsymbol{i}+\boldsymbol{k} \in \mathbb{N}^n} c_{\boldsymbol{i}} X_1^{i_1+k_1} \cdots X_n^{i_n+k_n} \quad \text{and} \quad G' = \sum_{\boldsymbol{j}+\boldsymbol{k} \in \mathbb{N}^n} d_{\boldsymbol{j}} X_1^{j_1+k_1} \cdots X_n^{j_n+k_n},$$

where $\boldsymbol{k} = (k_1, \ldots, k_n)$.

Finally, as a 1-semifield, K is cancellative under multiplication. Therefore if $F(b) \not\cong_\nu G(b)$ for some $0 \neq b \in K^n \setminus S$, then also $F'(b) \not\cong_\nu G'(b)$. This proves the second claim. $\square$

Since a line, most naturally, consists of pairs of real numbers, also the following lemma assumes the tangible points to be real numbers.

Lemma 6.22. *Let K be a 1-semifield such that $K_1 = \mathbb{T}$. A layered line, the slope of which is a rational number, is a congruence variety (in K^2).*

Proof. Let L be a line at the tangible layer K_1, when its equation can be given in the form

$$bY = aX + c,$$

where $a, b \in \mathbb{Z}$ and $c \in \mathbb{R}$. (Note that $\mathbb{R} \subset K_1$.)

Suppose first that a and b are non-negative. The above equation of L gives us two tropical polynomials

$$F = Y^b \quad \text{and} \quad G = cX^a,$$

which are elements of $K[X, Y]$.

Now, F and G reach ν-equivalent values at all the points of the layered line $L^{(\nu)}$, which means that $(F, G) \in \Omega_{L^{(\nu)}}$. Moreover, the points of $L^{(\nu)}$ are the only points, where F and G have ν-equivalent values. Therefore $L^{(\nu)} = V(\Omega_{L^{(\nu)}})$.

If a or b (or both) are negative, then $F, G \in K[X^{\pm 1}, Y^{\pm 1}]$. Based on Lemma 6.21, we can replace F and G with $F', G' \in K[X, Y]$ such that $(F', G') \in \Omega_{L^{(\nu)}}$ exactly when $(F, G) \in \Omega_{L^{(\nu)}}$. Therefore, we obtain again $L^{(\nu)} = V(\Omega_{L^{(\nu)}})$. □

Remark. The requirement that the slope of the line must be a rational number is needed, if only natural numbers are allowed as the exponents of the indeterminants in tropical polynomials. The same requirement is needed in the case of rational exponents, but the requirement excludes real exponents.

6.2.4 Layered ray and line segment

To show that a layered ray and line segment are congruence varieties, we need the following two lemmata.

Lemma 6.23. *Let K be a 1-semifield, $F \in K[X_1, \ldots, X_n]$, and $a \in K_1^n$. There exists an infinite number of monomials $G \in K[X_1, \ldots, X_n]$ such that the polynomial $F+G$ has a corner root at a. If moreover $F \in K_1[X_1, \ldots, X_n]$, then also $G \in K_1[X_1, \ldots, X_n]$.*

Proof. Suppose that H is a dominating monomial in F at a (perhaps among others), and write it in the form

$$H = cX_1^{d_1} \cdots X_n^{d_n},$$

where $c \in K$, and $d_1, \ldots, d_n \in \mathbb{N}$. By writing $a = (a_1, \ldots, a_n)$, it holds

$$F(a) \cong_\nu H(a) = cX_1^{d_1} \cdots X_n^{d_n}(a_1, \ldots, a_n) = ca_1^{d_1} \cdots a_n^{d_n}.$$

As an example of a monomial, different from H, we have

$$G = ca_1^{k_1} \cdots a_n^{k_n} X_1^{d_1-k_1} \cdots X_n^{d_n-k_n},$$

where $k_1, \ldots, k_n \in \mathbb{Z}$ such that $k_i \neq 0$, for some $i \in \{1, \ldots, n\}$ (to make sure that $G \neq H$). Choosing each k_i ($i \in \{1, \ldots, n\}$) to be negative ensures that $G \in K[X_1, \ldots, X_n]$. With such a choice, each $a_i^{k_i}$ has a negative exponent, but it does not matter, since $a_i \in K_1$ for all $i \in \{1, \ldots, n\}$ and K is a 1-semifield.

Now, it holds that

$$G(a) = ca_1^{k_1} \cdots a_n^{k_n} a_1^{d_1-k_1} \cdots a_n^{d_n-k_n} = ca_1^{d_1} \ldots a_n^{d_n} = H(a).$$

Therefore, at least two monomials in $F + G$ has the same value at a, and thus, $F + G$ has a corner root at a. Note that the exponents $k_1, \ldots, k_n$ are elements of $\mathbb{Z}$ (or $\mathbb{Z}_-$). By varying them, we can find an infinite number of monomials G.

The last assertion is clear. □

Lemma 6.24. *Let K be an infinite 1-semifield, $F \in K[X_1, \ldots, X_n]$, and $a, b \in K_1^n$ distinct tangible points, i.e. $a \neq b$. If F has a monomial dominating both at a and b, then there exists a monomial $G \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ such that*

$$(F + G)(a) \cong_\nu F(a) \quad \textit{and} \quad (F + G)(b) >_\nu F(b).$$

Moreover, if $F \in K_1[X_1, \ldots, X_n]$, then also $G \in K_1[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

Proof. Based on Lemma 6.23, there exists an infinite number of monomials G such that $F + G$ has a corner root at a. Therefore, $(F + G)(a) \cong_\nu F(a)$ with an infinite number of monomials G. Hence, we can always find a monomial G that is different from the original monomials in F. To show that it is possible to choose G such that $(F + G)(b) >_\nu F(b)$, we need K to be infinite to ensure that there is a sufficient number of values for finding a monomial dominating all the original monomials in F at b.

Let H denote a monomial in F dominating both at a and b, and write it as

$$H = cX_1^{d_1} \cdots X_n^{d_n},$$

where $c \in K$ and $d_1, \ldots, d_n \in \mathbb{N}$. Write $a = (a_1, \ldots, a_n)$, when based on the proof of Lemma 6.23, the additional monomial G is of the form

$$G = ca_1^{k_1} \cdots a_n^{k_n} X_1^{d_1-k_1} \cdots X_n^{d_n-k_n},$$

for $k_1, \ldots, k_n \in \mathbb{Z}$ such that $k_i \neq 0$, for some $i \in \{1, \ldots, n\}$.

We will next consider the values of G and H at $b = (b_1, \ldots, b_n)$. Now, $b \neq a$, and thus, there exists $b_i \neq a_i$ for some $i \in \{1, \ldots, n\}$. By choosing a positive k_i for each $b_i <_\nu a_i$ and by choosing a negative k_i for each $b_i >_\nu a_i$, we obtain

$$\begin{aligned} G(b) &= ca_1^{k_1} \cdots a_n^{k_n} b_1^{d_1-k_1} \cdots b_n^{d_n-k_n} = ca_1^{k_1} b_1^{d_1-k_1} \cdots a_n^{k_n} b_n^{d_n-k_n} \\ &= cb_1^{d_1}(a_1 b_1^{-1})^{k_1} \cdots b_n^{d_n}(a_n b_n^{-1})^{k_n} >_\nu cb_1^{d_1} b_n^{d_n} = H(b). \end{aligned}$$

Namely, the above made choice implies that $(a_i b_i^{-1})^{k_i} >_\nu 1_K$ for all $i \in \{1, \ldots, n\}$. Note also that $b_i^{-1} \in K$ for all $i \in \{1, \ldots, n\}$, since K is a 1-semifield and $b \in K_1$. When we choose k_i to be positive or negative (based on the values a_i and b_i), we cannot avoid the exponents $d_i - k_i$ from becoming negative, unless the original exponents d_i are great enough. Therefore, $G \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

Recall that H is a monomial in F dominating also at b. Therefore

$$F(b) \cong_\nu H(b) <_\nu G(b),$$

which implies $F(b) + G(b) \cong_\nu G(b)$. Hence,

$$F(b) <_\nu G(b) \cong_\nu F(b) + G(b) = (F + G)(b).$$

The last assertion is clear. □

Remark. If all the original exponents $d_1, \ldots, d_n$ in F (and thus, those in H) are greater or equal to 1, we can find $G \in K[X_1, \ldots, X_n]$. Namely, when we need to choose k_i to be positive, we can take $k_i = d_i$. This is possible, since $d_i \geq 1$. In this way, we have $d_i - k_i = 0$. Note that we have to select G to be different from the existing monomials of F. The selection described above is safe, since we assumed F to have no indeterminant with the zero exponent.

In the same way, as when proving a line to be congruence variety (Lemma 6.22), the following lemma assumes the tangible values to be real numbers. This assumption also makes the 1-semifield infinite, as required in Lemma 6.24.

Lemma 6.25. *Let K be a 1-semifield such that $K_1 = \mathbb{T}$. A layered ray and a layered line segment, both with rational endpoints, are congruence varieties (in K^2).*

Proof. We will first concentrate on proving the claim in the case of a ray. This is done by presenting certain actions to be made at the endpoint of it.

Let L be a tangible line and S a tangible ray such that $S \subset L \subset K_1^2$ and $S^{(\nu)} \subset L^{(\nu)} \subset K^2$. Based on Lemma 6.22, $L^{(\nu)}$ is a congruence variety. Based on the proof of the same lemma, there exists two monomials, $F_1, F_2 \in K[X, Y]$ reaching ν-equivalent values exactly at the points of $L^{(\nu)}$. More precisely,

$$x \in L^{(\nu)} \iff F_1(x) \cong_\nu F_2(x).$$

Denote $F = F_1 + F_2$. Suppose that a is the endpoint of S, when $a \in S \subset L$. Take another point $b \in L \backslash S$, when clearly $a \neq b$. Since $a, b \in L \subset L^{(\nu)}$, both F_1 and F_2 are such monomials in F that dominate both at a and b. Therefore, based on Lemma 6.24, there exist a monomial $G \in K[X^{\pm 1}, Y^{\pm 1}]$ such that

$$(F+G)(a) \cong_\nu F(a) \quad \text{and} \quad (F+G)(b) >_\nu F(b).$$

Based on the proofs of Lemmata 6.23 and 6.24, it actually holds that $G(a) \cong_\nu F(a)$. Therefore, all three monomials in $F+G$ dominate at a, i.e.

$$F_1(a) \cong_\nu F_2(a) \cong_\nu G(a).$$

(An example of such a situation is depicted in Figure 2.3.)

Based on Lemma 6.21, we can assume that $G \in K[X, Y]$, since otherwise we could replace F_1, F_2 and G with $F_1', F_2', G' \in K[X, Y]$, respectively, such that these monomials have mutually ν-equivalent values exactly at the same points as the original monomials have.

Since the polynomial $F+G$ consists of three monomials (that are all different from each other), there are three tropical regions, where each monomial dominates. Clearly, F_1 and F_2 dominate together $F+G$ exactly at the points of $S^{(\nu)}$. More precisely,

$$x \in S^{(\nu)} \quad \Longleftrightarrow \quad F_1(x) \cong_\nu F_2(x) \geq_\nu G(x).$$

Therefore if $x \in S^{(\nu)}$, then

$$G(x) \leq_\nu F_1(x) + F_2(x) = (F_1 + F_2)(x) = F(x).$$

This can be formulated as $F(x) \cong_\nu (F+G)(x)$, for all $x \in S^{(\nu)}$, which means that $(F, F+G) \in \Omega_{S^{(\nu)}}$. Clearly, also $(F_1, F_2) \in \Omega_{S^{(\nu)}}$, due to the fact that $(F_1, F_2) \in \Omega_{L^{(\nu)}}$

When calculating $V(\Omega_{S^{(\nu)}})$, we search for the points, where the components of all pairs in $\Omega_{S^{(\nu)}}$ are ν-equivalent with each other. As just shown, $(F_1, F_2), (F, F+G) \in \Omega_{S^{(\nu)}}$, and when finding the common points, where especially these pairs agree, we find exactly the points of $S^{(\nu)}$. Namely, (F_1, F_2) agree at the points of $L^{(\nu)}$, and $(F, F+G)$ agree at the points of $\overline{D}_{F+G,1} \cup \overline{D}_{F+G,2}$, where the indices 1 and 2 refer to F_1 and F_2, respectively. The common points of these two sets are exactly the points of $S^{(\nu)}$. Therefore, $S^{(\nu)} = V(\Omega_{S^{(\nu)}})$.

A line segment can be expressed as an intersection of two rays included in the same line. Since a ray is a congruence variety (as just proved), Lemma 6.10 implies that a line segment is a congruence variety. (Alternatively, we can repeat the actions described above for both of the endpoints of a line segment.) $\square$

The following example clarifies Lemma 6.25.

Example 6.26. Let $R = \mathcal{R}(\{1\}, \mathbb{T})$, and take the points $(2,3), (1,1) \in R^2$ as the endpoints of a line segment, which is denoted by S. Furthermore, denote the line determined by these points as L. Now, $2X^2, 3Y \in R[X,Y]$ are such monomials that agree exactly at the points of L. Based on Lemma 6.21, we can multiply both of them by Y, as well as by any monomial, in order to avoid negative exponents in sequel. This gives us monomials F_1 and F_2, as follows

$$F_1 = 2X^2Y \quad \text{and} \quad F_2 = 3Y^2.$$

We can now find monomial G based on F_1 and the endpoint $(1,1)$. When searching for an additional point that lies on L but outside S, we can take, for example, $(0,-1)$. (Any x-coordinate less than 1 goes.) Based on the proof of Lemma 6.24, we can construct G as

$$G = 2 \odot 1^k \odot 1^l \odot X^{2-k} \odot Y^{1-l},$$

where the first factor comes from the coefficient of F_1, the second one from the x-coordinate of $(1,1)$, and the third one from the y-coordinate of the same point.

Since the x-coordinate of $(0,-1)$ is less than that of $(1,1)$, we can select $k = 1$ (positive). Since the same holds for y-coordinates, we can select also $l = 1$ (positive). These choices give

$$G = 2 \odot 1^1 \odot 1^1 \odot X^{2-1} \odot Y^{1-1} = 4X.$$

Consider next the other endpoint of the line segment. We determine another additional monomial H based on F_2 and the endpoint $(2,3)$. A suitable additional point is now $(3,5)$. Therefore

$$H = 3 \odot 2^k \odot 3^l \odot X^{0-k} \odot Y^{2-l},$$

where the first factor comes from the coefficient of F_2, the second one from the x-coordinate of $(2,3)$, and the third one from the y-coordinate of the same point. Now, y-coordinate of $(3,5)$ is greater than that of $(2,3)$, and thus, we can select $k = -1$ (negative). The same holds for y-coordinates, and thus, we can select also $l = -1$ (negative). These choices give

$$H = 3 \odot 2^{-1} \odot 3^{-1} \odot X^{0-(-1)} \odot Y^{2-(-1)} = -2XY^3.$$

As desired, the monomials $F_1 = 2X^2Y$, $F_2 = 3Y^2$ and $G = 4X$ dominate at $(1,1)$, and the monomials $F_1 = 2X^2Y$, $F_2 = 3Y^2$ and $H = -2XY^3$ dominate at $(2,3)$. Moreover, G strictly dominates $F_1 \oplus F_2$ at those points that are downwards the line segment, and H strictly dominates $F_1 \oplus F_2$ at those points that are upwards the line segment. To see this more clearly, we can calculate the regions, where each monomial dominates the others. The result of this calculation is presented in Figure 6.1.

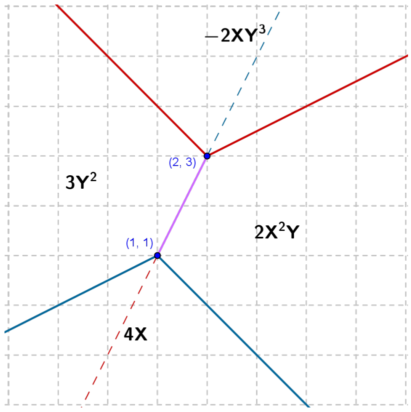


Figure 6.1: The corner loci of the polynomials $2X^2Y \oplus 3Y^2 \oplus 4X$ (drawn in blue and violet) and $2X^2Y \oplus 3Y^3 \oplus (-2)XY^3$ (drawn in red and violet).

The pairs

$$\begin{aligned}
&(2X^2Y, 3Y^2),\\
&(2X^2Y \oplus 3Y^2, 2X^2Y \oplus 3Y^2 \oplus 4X),\\
&(2X^2Y \oplus 3Y^2, 2X^2Y \oplus 3Y^2 \oplus (-2)XY^3)
\end{aligned}$$

are elements of Ω_S. Moreover, when searching for the points, where all three pairs above reach ν-equivalent values, we find exactly the points of S. Hence, $S = V(\Omega_S)$.

6.2.5 Closed tropical region

As the final example of a congruence variety, we consider a closed tropical region.

Lemma 6.27. *A closed tropical region is a congruence variety.*

Proof. Let K be a 1-semifield. Consider an arbitrary closed tropical region, $\overline{D}_{F,i}$, where $F \in K[X_1, \dots, X_n]$ and i refers to the ith monomial in F, which is denoted as F_i.

Directly based on the definition of a closed tropical region, $(F, F_i) \in \Omega_{\overline{D}_{F,i}}$. Since the points of $\overline{D}_{F,i}$ are the only points, where F and F_i are ν-equivalent, then also $\overline{D}_{F,i} = V(\Omega_{\overline{D}_{F,i}})$. Hence, $\overline{D}_{F,i}$ is a congruence variety. $\square$

Remark. It is possible that $\overline{D}_{F,i} = \emptyset$ or $\overline{D}_{F,i} = K^n$, but Example 6.3 has shown that the empty set and the whole space can be congruence varieties. Such cases occur, respectively, if F_i is inessential in F, or if F is a monomial.

6.3 Congruence varieties related to algebraic sets

This section shows the correspondence between algebraic sets and congruence varieties. The relationship is described in the following two lemmata.

Lemma 6.28. *An algebraic set is a congruence variety, when taking only tangible roots into account.*

Proof. Let K be a 1-semifield and $I \subset K[X_1, \ldots, X_n]$ a subset. It is required to show that $\mathcal{Z}(I) = V(\Omega_{\mathcal{Z}(I)})$.

Recall that
$$\mathcal{Z}(I) = \bigcap_{F \in I} \mathcal{Z}(F).$$
Based on Lemma 5.15, $\mathcal{Z}(F)$ can be expressed by finite unions and intersections of closed tropical regions (when taking only tangible roots into account). Based on Lemma 6.27, each of these closed tropical regions is a congruence variety. Finally, based on Lemma 6.12 (as well as Lemma 6.13), a finite union of congruence varieties is again a congruence variety, and based on Lemma 6.10 (as well as Lemma 6.11), an intersection of congruence varieties is again a congruence variety.

Until now we have proved $\mathcal{Z}(F)$ to be a congruence variety. Lemma 6.10 (as well as Lemma 6.11) can be applied also for infinite intersections of congruence varieties, and thus, $\mathcal{Z}(I)$ is a congruence variety.

Hence, a total algebraic set is a congruence variety. □

Remark. Note that the above claim holds true also for corner algebraic sets, since they can be expressed by the above kind sets, too.

Next lemma proves the claim opposite for the previous lemma. In other words, it shows that a congruence variety is an algebraic set. We will later see that this does not hold for a corner algebraic set. Namely, cluster roots are crucial in the proof of the lemma, and thus, the sorting semiring† is required to include at least two elements.

Lemma 6.29. *A congruence variety that is a tangible subset of a 1-semifield with at least two layers is a total algebraic set (i.e. only tangible roots are taken into account).*

Proof. Let K be a 1-semifield with at least two layers. Consider an arbitrary tangible congruence variety $X \subset K_1$, when $X = V(\Omega_X)$. We claim that
$$X = \mathcal{Z}(I),$$
where
$$I := \{F + G = H \in K[X_1, \ldots, X_n] \mid (F, G) \in \Omega_X\}.$$

"$\subset$" Suppose that $a \in X = V(\Omega_X)$. Therefore $F(a) \cong_\nu G(a)$ for all $(F,G) \in \Omega_X$. Now, at a, as at any other point, both F and G have one or several monomials giving the maximum value. Suppose first that both F and G have a single dominating monomial at a and these monomials are the same. Denote such a common monomial as $cX_1^{i_1} \cdots X_n^{i_n}$, where $c \in K$. When calculating $F+G$, we calculate also

$$cX_1^{i_1} \cdots X_n^{i_n} + cX_1^{i_1} \cdots X_n^{i_n} = (c+c)X_1^{i_1} \cdots X_n^{i_n},$$

where the coefficient $c + c \in K_{>1}$, and thus, $(c+c)X_1^{i_1} \cdots X_n^{i_n}(a) \in K_{>1}$. This means that a is a cluster root of $F+G$, and thus, $a \in \mathcal{Z}(H)$.

If either F or G has several dominating monomials, or both have one that are not the same, then $F+G$ has several monomials dominating at a. This means that a is a corner root of $F+G$, and thus, $a \in \mathcal{Z}_{\text{corn}}(H) \subset \mathcal{Z}(H)$.

(It is possible that both F and G has the same dominating monomial, but besides it, either F or G or both have other dominating monomials, too. In such a case, the common monomials are added together, when they become a single monomial in $F+G$, as $(c+c)X_1^{i_1} \cdots X_n^{i_n}$ above. Since there exists also other dominating monomials, a is a corner root.)

Since the condition $F(a) \cong_\nu G(a)$ holds for all $(F,G) \in \Omega_X$, a is (either a cluster or corner) root of all $F+G = H$. Hence, $a \in \mathcal{Z}(I)$.

"$\supset$" We apply contraposition and suppose that a tangible point $a \in K_1$ such that $a \notin V(\Omega_X)$. This means that $F(a) \ncong_\nu G(a)$, for some $(F,G) \in \Omega_X$. We can assume that $F(a) <_\nu G(a)$. Therefore a is a root of $F+G$, exactly when it is a root G. We will show that even if a is a root of $F+G$ (and $F+G \in I$), it does not hold that $a \in \mathcal{Z}(I)$.

If a is a cluster root of G, we apply tangible lift to the coefficient of each monomial in G. In this way, we have $(F,\hat{G}) \in \Omega_X$ such that

$$F(a) <_\nu G(a) \cong_\nu \hat{G}(a).$$

Since a is tangible and all the coefficients of $\hat{G}$ are tangible, a is not a cluster root (nor a corner root) of $\hat{G}$. Therefore a is not a cluster root (nor a corner root) of $F+\hat{G}$, although $(F,\hat{G}) \in \Omega_X$, implying that $F+\hat{G} \in I$. Hence $a \notin \mathcal{Z}(I)$.

If a is a corner root of G, we can again suppose that all the coefficients of G are tangible. Otherwise, we could apply tangible lift to the coefficient of each monomial in G, as in the first case. Now G has at least two monomials dominating at a. We will consider two opposite situations.

Assume first that at least one of those monomials in G that dominate at a, dominates also at some point of X. Note that we can decompose X into subsets based on the dominating monomials in G. More precisely, if G_i is a monomial in G dominating at a subset of X, we denote this subset as X_i. (Such subsets need not be disjoint.) Now based on the just made assumption, there is a monomial G_j in G dominating both at the subset X_j and at the point a.

Since $(F, G) \in \Omega_X$, we have

$$F(x) \cong_\nu G(x) \cong_\nu G_j(x)$$

for all $x \in X_j$. Therefore,

$$F(x) \cong_\nu (F + G_j)(x)$$

for all $x \in X_j$. If we denote $G' = F + G_j$, we have $(F, G') \in \Omega_{X_j}$. If $X_j = X$, we have already concluded that $(F, G') \in \Omega_X$. Otherwise, we need to consider also those parts of X, where G_j does not dominate. This means the set $X \backslash X_j$, which is now non-empty. If $x \in X \backslash X_j$, then

$$G'(x) = (F + G_j)(x) = F(x) + G_j(x) = F(x).$$

Namely, $(F, G) \in \Omega_X$, and therefore if G_j does not dominate G at x, it does not dominate F at x either. We have now concluded that $(F, G') \in \Omega_{X \backslash X_j}$. By combining the results achieved until now, we have $(F, G') \in \Omega_{X_j} \cap \Omega_{X \backslash X_j}$. Based on Proposition 6.5, this means that $(F, G') \in \Omega_{X_j \cup X \backslash X_j} = \Omega_X$.

Moreover,

$$(F + G')(a) = (F + F + G_j)(a) = F(a) + F(a) + G_j(a) = G_j(a) \notin K_{>1},$$

since G_j is a monomial in G dominating at a and $F(a) <_\nu G(a)$. Now, $G_j(a)$ is tangible, since both a and the coefficient of G_j are tangible. Therefore a is not a (corner) root of $F + G'$, although $(F, G') \in \Omega_X$. Hence, $a \notin \mathcal{Z}(I)$. (Note that G_j need not be a single monomial dominating at X_j and we have not assumed such in the above reasoning.)

Consider finally the opposite situation, where none of those monomials in G that dominate at a, dominate at any point of X. (Point a is still assumed to be a corner root of G.) In this case, we can construct G', based on G, by keeping one of these monomials that dominate at a and dropping all the other such monomials away. More precisely, if G has n monomials ($n \geq 2$) dominating at a, we take one of them into G' and leave the other $n - 1$ monomials without taking. Other kind of monomials in G (that do not dominate at a) are taken into G'. After this action there is no difference on how G and G' behave at X (up to ν-equivalence), and thus, we have $G'(x) \cong_\nu G(x)$ for all $x \in X$. Therefore $F(x) \cong_\nu G'(x)$ for all $x \in X$, which means that $(F, G') \in \Omega_X$. Moreover,

$$F(a) <_\nu G(a) \cong_\nu G'(a).$$

Now $G'(a) \notin K_{>1}$, since there is only one such a monomial left that dominates at a. Therefore

$$(F + G')(a) = F(a) + G'(a) = G'(a) \notin K_{>1}.$$

This means that a is not a (corner) root of $F + G'$, although $(F, G') \in \Omega_X$. Hence, $a \notin \mathcal{Z}(I)$. $\square$

Remark. Cluster roots are crucial in the above lemma. Namely, based on Lemma 6.18 each closed interval is a congruence variety. However, in the case of a polynomial in one indeterminant, the set of corner roots cannot be an interval. Otherwise, such a polynomial would have an infinite number of corner roots, which requires that the polynomial should have an infinite number of monomials. In other words, a congruence variety is not necessarily a corner algebraic set.

The next examples clarify Lemma 6.29.

Example 6.30. Consider the 1-semifield $R := \mathcal{R}(\mathbb{N}^*, \mathbb{T})$ and the congruence variety $X := \{(1, 4)\} \subset R^2$. (If a layer value is 1, we do not write it visible.)

For example,

$$F = X \oplus Y \oplus 0 \quad \text{and} \quad G = (-5)X^2 \oplus (-5)Y^2 \oplus (-1)XY,$$

elements of $R[X, Y]$, form a pair of polynomials such that $(F, G) \in \Omega_X$. Namely,

$$\begin{aligned} &F(1, 4) = 1 \oplus 4 \oplus 0 = 4, \\ &G(1, 4) = (-5 \odot 2) \oplus (-5 \odot 8) \oplus (-1 \odot 1 \odot 4) = 4, \end{aligned}$$

and thus, $(1, 4)$ is a corner root of $F \oplus G$. This can also be seen in Figure 2.4, which depicts the sum polynomial $F \oplus G$.

We will next clarify the other direction of Lemma 6.29. As can be seen in Figure 2.4, there are several values $a \in \mathcal{Z}(F \oplus G)$. Take, for example, $a := (6, 2) \notin X$. This is a root of $F \oplus G$, since

$$(F \oplus G)(6, 2) = 6 \oplus 2 \oplus 0 \oplus (-5 \odot 12) \oplus (-5 \odot 4) \oplus (-1 \odot 6 \odot 2) = {}^{[2]}7.$$

However, $F(6, 2) <_\nu G(6, 2)$, and thus, we are not ready yet.

Those monomials in G that dominate at $(6, 2)$ are $-5X^2$ and $-1XY$. The latter one of these, $-1XY$, dominates also at $(1, 4)$. This means that we can find G' by adding this monomial to F. Therefore,

$$G' = F \oplus (-1)XY = X \oplus Y \oplus 0 \oplus (-1)XY,$$

when we have

$$(F \oplus G')(6, 2) = 6 \oplus 2 \oplus 0 \oplus 6 \oplus 2 \oplus 0 \oplus 7 = 7 \notin R_{>1},$$

which shows that $(6, 2)$ is not a root of $F \oplus G'$. We still have

$$G'(1, 4) = 1 \oplus 4 \oplus 0 \oplus 4 = {}^{[2]}4 \cong_\nu 4 = F(1, 4),$$

which means that $(F, G') \in \Omega_X$. Therefore, $F \oplus G' \in I$, but $a \notin \mathcal{Z}(F \oplus G')$, when also $a \notin \mathcal{Z}(I)$.

Example 6.31. Consider the 1-semifield $R := \mathcal{R}(\mathbb{N}^*, \mathbb{T})$ and the congruence variety $X := \{(5,0)\} \subset R^2$. (If a layer value is 1, we do not write it visible.)

We can take the same polynomials as in the previous example,

$$F = X \oplus Y \oplus 0 \quad \text{and} \quad G = (-5)X^2 \oplus (-5)Y^2 \oplus (-1)XY,$$

as such elements of $R[X,Y]$ that $(F,G) \in \Omega_X$. Clearly, $(5,0)$ is a root of $F \oplus G$, since

$$(F \oplus G)(5,0) = 5 \oplus 0 \oplus 0 \oplus (-5 \odot 10) \oplus (-5 \odot 0) \oplus (-1 \odot 5 \odot 0) = {}^{[2]}5.$$

This can again be seen in Figure 2.4, too.

Take $a := (2,6) \notin X$. In the same way as earlier, a is a root of $F \oplus G$, but

$$F(2,6) = 6 <_\nu {}^{[2]}7 = G(2,6),$$

and thus, we are not ready yet.

Those monomials in G that dominate at $(2,6)$ are $-5Y^2$ and $-1XY$. Neither of them dominates at $(5,0)$. This means that we can find G' by removing either one of these monomials from G. If we remove the latter one, we have $G' = -5X^2 \oplus (-5)Y^2$. In this way, we obtain

$$(F \oplus G')(2,6) = 2 \oplus 6 \oplus 0 \oplus (-5 \odot 4) \oplus (-5 \odot 12) = 7 \notin R_{>1},$$

but still

$$G'(5,0) = (-5 \odot 10) \oplus (-5 \odot 0) = 5 = F(5,0),$$

and thus, $F \oplus G' \in I$, but $a \notin \mathcal{Z}(F \oplus G')$, when also $a \notin \mathcal{Z}(I)$.

Example 6.32. Consider the 1-semifield $R := \mathcal{R}(\mathbb{N}^*, \mathbb{T})$ and the congruence variety

$$X := \{(1-t)(5,0) + t(5,-1) \mid t \in [0,1]\}.$$

(If a layer value is 1, we do not write it visible.) Although X is a line segment instead of a point, there is no big difference to the previous examples. Namely, we again take the same polynomials,

$$F = X \oplus Y \oplus 0 \quad \text{and} \quad G = (-5)X^2 \oplus (-5)Y^2 \oplus (-1)XY$$

as such elements of $R[X,Y]$ that $(F,G) \in \Omega_X$. All the points of X are roots of $F \oplus G$, which can be easily seen in Figure 2.4.

To clarify the other direction of Lemma 6.29, take first $a := (6,2) \notin X$. Based on earlier examples, this is a root of $F \oplus G$, but $F(6,2) <_\nu G(6,2)$. Those monomials in G that dominate at $(6,2)$ are $-5X^2$ and $-1XY$. The first one of these, $-5X^2$, dominates also at X. This means that we can find G' by adding this monomial to F. Therefore, $G' = X \oplus Y \oplus 0 \oplus (-5)X^2$, when we have

$$(F \oplus G')(6,2) = 6 \oplus 2 \oplus 0 \oplus 6 \oplus 2 \oplus 0 \oplus 7 = 7 \notin R_{>1}.$$

Clearly, we still have $G'(x) \cong_\nu F(x)$ for all $x \in X$.

Suppose alternatively that we take $a := (2,6) \notin X$. Now again, this is a root of $F \oplus G$, but $F(2,6) <_\nu G(2,6)$. Those monomials in G that dominate at $(2,6)$ are $-5Y^2$ and $-1XY$. Neither of them dominates at X. This means that we can find G' by dropping either of these monomials away from G. If we drop the first one, we have $G' = -5X^2 \oplus (-1)XY$. Now,

$$(F \oplus G')(2,6) = 2 \oplus 6 \oplus 0 \oplus (-1) \oplus 7 = 7 \notin R_{>1},$$

but we still have $G'(x) \cong_\nu G(x) \cong_\nu F(x)$ for all $x \in X$.

Example 6.33. Consider the 1-semifield $R := \mathcal{R}(\mathbb{N}^*, \mathbb{T})$ and the congruence variety $X := \{(1,0)\} \subset R^2$. (If a layer value is 1, we do not write it visible.) We can take

$$F = X \oplus Y \oplus 0 \quad \text{and} \quad G = X \oplus (-5)X^2 \oplus (-5)Y^2 \oplus (-1)XY$$

as such elements of $R[X,Y]$ that $(F,G) \in \Omega_X$. (Note that G is slightly different polynomial from that of the previous examples.) In this case,

$$F \oplus G = {}^{[2]}0X \oplus Y \oplus 0 \oplus (-5)X^2 \oplus (-5)Y^2 \oplus (-1)XY.$$

Now, $F(1,0) = G(1,0)$, and $(1,0)$ is also a root of $F \oplus G$. Namely,

$$(F \oplus G)(1,0) = {}^{[2]}1 \oplus 0 \oplus 0 \oplus (-5 \odot 2) \oplus (-5 \odot 0) \oplus (-1 \odot 1 \odot 0) = {}^{[2]}1.$$

More precisely, $(1,0)$ is a cluster root of $F \oplus G$.

Take $a := (2,6) \notin X$, which is a root of $F \oplus G$. The monomials in G dominating at $(2,6)$ are $-5Y^2$ and $-1XY$. Neither of them dominates at $(1,0)$. This means that we can find G' by removing e.g. the latter one from G, when we obtain $G' = X \oplus (-5)X^2 \oplus (-5)Y^2$. In this way, it holds that

$$(F \oplus G')(2,6) = 2 \oplus 6 \oplus 0 \oplus 2 \oplus (-5 \odot 4) \oplus (-5 \odot 12) = 7 \notin R_{>1},$$

but still

$$G'(1,0) = 1 \oplus (-5 \odot 2) \oplus (-5 \odot 0) = 1 = F(1,0),$$

and thus, $F \oplus G' \in I$, but $a \notin \mathcal{Z}(I)$.

Consider finally otherwise the same situation, but the polynomials declared above are now elements of $\mathbb{T}[X,Y]$. In this case, the first direction does not work, since $(1,0) \in X$ and $F \oplus G \in I$, but $(1,0)$ is not a root of $F \oplus G$, since now

$$F \oplus G = X \oplus Y \oplus 0 \oplus (-5)X^2 \oplus (-5)Y^2 \oplus (-1)XY,$$

when X is the only monomial in $F \oplus G$ dominating at $(1,0)$.

However, the second direction would be exactly the same as above, since there are no ν-equivalences to be replaced with an equation.

The next example clarifies the similarity between algebraic sets and congruence varieties.

Example 6.34. Based on Examples 5.2 and 6.3, both an algebraic set and a congruence variety can be the empty set, as well as the whole space.

Based on Example 5.3, a layered point is an algebraic set, and based on Lemma 6.16, a layered point is a congruence variety.

Similarly, Lemmata 6.22 and 6.25 show that a layered line, a layered ray, and a layered line segment are congruence varieties. Based on Examples 5.4 and 5.5 with the remark after the latter one, we can easily see that the aforementioned sets are also algebraic sets.

6.4 Algebraic sets modulo a congruence

The previous section show that in the case of several layers, each congruence variety is an algebraic set, and each algebraic set is a congruence variety, when taking only tangible roots into account. In other words, congruence varieties and algebraic sets can be expressed in the terms of each other. This section introduces a third kind of set that can be expressed in the terms of both of the aforementioned two sets, and vice versa. As in the previous section and especially in Lemma 6.29, we will assume again the sorting semiring† to include at least two elements.

We start with the following definitions.

Definition 6.35. Let K be a 1-semifield with at least two layers, $S \subset K^n$ a subset, Ω a congruence of S on $K[X_1, \ldots, X_n]$ and $F, G \in K[X_1, \ldots, X_n]$. The *total locus of* (F, G) *modulo* Ω *(with respect to* S*)* is

$$\mathcal{Z}((F, G); S)_\Omega := \{a \in S \mid F(a) \equiv G(a) \in K_{>1}\}.$$

If $S = K^n$, we write $\mathcal{Z}((F, G))_\Omega$ instead of $\mathcal{Z}((F, G); K^n)_\Omega$.

Remark. Since we require the sorting semiring† to have at least two elements, Lemma 4.31 implies that the condition $G(a) \in K_{>1}$ holds true, exactly when a is a root of G.

Definition 6.36. Let K be a 1-semifield with at least two layers, $S \subset K^n$ a subset, Ω a congruence of S on $K[X_1, \ldots, X_n]$, $F, G \in K[X_1, \ldots, X_n]$, and $A \subset K[X_1, \ldots, X_n] \times K[X_1, \ldots, X_n]$ a non-empty subset. The *(affine) (total) algebraic set of* A *modulo* Ω *(with respect to* S*)* is

$$\mathcal{Z}(A; S)_\Omega := \bigcap_{(F,G) \in A} \mathcal{Z}((F, G); S)_\Omega.$$

If $S = K^n$, we write $\mathcal{Z}(A)_\Omega$ instead of $\mathcal{Z}(A; K^n)_\Omega$.

Example 6.37. Let $K := \mathcal{R}(\mathbb{N}^*, \mathbb{T})$, a uniform (layered) 1-semifield, and $F \in K[X_1, \dots, X_n]$, written as $F = \sum_{i=1}^{r} F_i$. If the coeffient of F_i, for some $i \in \{1, \dots, r\}$, is ghost-valued, then the closed tropical region $\overline{D}_{F,i}$ is an algebraic set modulo a congruence, where the congruence in question is ν-equivalence.

In other words,

$$\overline{D}_{F,i} = \mathcal{Z}((F, F_i))_{\cong_\nu}.$$

We have earlier shown the connection between a congruence variety and an algebraic set (in Lemmata 6.28 and 6.29). We will next present a similar kind of connection between each of these sets and an algebraic set modulo a congruence.

Lemma 6.38. *An algebraic set is an algebraic set modulo ν-equivalence.*

Proof. Let K be a 1-semifield (with at least two layers), $I \subset K[X_1, \dots, X_n]$, and

$$A := \{(F, F) \mid F \in I\}.$$

Then clearly

$$\bigcap_{F \in I} \mathcal{Z}(I) = \bigcap_{(F,F) \in A} \mathcal{Z}(A)_{\cong_\nu}.$$

$\square$

Lemma 6.39. *An algebraic set modulo ν-equivalence is a congruence variety, when taking only tangible roots into account.*

Proof. Let K be a 1-semifield with at least two layers, and suppose that $F, G \in K[X_1, \dots, X_n]$. Based on Definition 6.35, it is clear that

$$\mathcal{Z}((F, G))_{\cong_\nu} = \mathcal{Z}(G) \cap V(\{(F, G)\}).$$

Namely $a \in \mathcal{Z}((F, G))_{\cong_\nu}$ exactly when the condition $F(a) \cong_\nu G(a) \in K_{>1}$ holds true. This condition can be decomposed into two parts: $F(a) \cong_\nu G(a)$ and $G(a) \in K_{>1}$. The former part holds true, if and only if $a \in V(\{(F, G)\})$, while the latter part, in the case of several layers, is equivalent to $a \in \mathcal{Z}(G)$, as proved in Lemma 4.31.

To show that the same thing holds true more generally, suppose that $A \subset K[X_1, \dots, X_n] \times K[X_1, \dots, X_n]$, and define

$$I := \{G \in K[X_1, \dots, X_n] \mid (F, G) \in A \text{ for some } F \in K[X_1, \dots, X_n]\}.$$

Then clearly the above result holds true also in the form

$$\bigcap_{(F,G) \in A} \mathcal{Z}(A)_{\cong_\nu} = \bigcap_{G \in I} \mathcal{Z}(I) \cap \bigcap_{(F,G) \in A} V(A).$$

The claim follows, when recalling (from Lemma 6.9) that $V(A) = V(\langle A \rangle)$, and (from Lemma 6.28) that each algebraic set is a congruence variety when taking only tangible roots into account, and (from Lemma 6.10 or 6.11) that an intersection of congruence varieties is a congruence variety. $\square$

Based on the above two lemmata, we know that all the sets:

(i) an algebraic set

(ii) an algebraic set modulo ν-equivalence

(iii) a congruence variety

are the same or can be expressed in the terms of each other, when taking only tangible roots into account and when the sorting semiring† in question has several layers.

The lemmata, based on which the above connection realizes, have the restriction "when taking only tangible roots into account". We will next speculate how to get rid of it. Consider first the following fundamental properties of congruence varieties and algebraic sets.

(FP-C) If a is an element of a congruence variety, then all elements that are ν-equivalent to a are also the elements of the congruence variety.

(FP-A) If a is an element of an algebraic set, then all elements that are ν-equivalent to a are also the elements of the algebraic set.

Based on Proposition 4.13, it is easy to see that (FP-C) holds true for congruence varieties. However, (FP-A) holds true for corner roots, but not for cluster roots. Namely, if R is a layered semiring and $a \in R_{>1}$, then it is possible that a is a root of a polynomial, although $\hat{a}$ is not.

To improve this situation, we give an alternative definition for a root. Recall from Definition 4.29 and Lemma 4.31 that for a layered semiring† R with at least two layers, it holds that $a \in R$ is a root of polynomial F over R, if $F(a) \in R_{>1}$. The following alternative way to define a root is not far away from this.

Definition 6.40 (Speculative)**.** Let R be a layered semiring† with at least two layers and $F \in R[X_1, \dots, X_n]$. It is said that $a \in R$ is a *root* of F, if $F(\hat{a}) \in R_{>1}$.

Remark. The truth value of condition $F(\hat{a}) \in R_{>1}$ does not depend on the selection of $\hat{a}$. To see this, let $a_1, a_2 \in R_1$ such that $a \cong_\nu a_1$ and $a \cong_\nu a_2$, but $a_1 \neq a_2$. Now $a_1 \cong_\nu a_2$, based on the transitive property of ν-equivalence.

We will show that $F(a_1) \in R_{>1}$ implies $F(a_2) \in R_{>1}$. Suppose that $F(a_1) \in R_{>1}$. If there is a single monomial in F dominating at a_1, then the coefficient of the monomial must be ghost-valued, since otherwise the assumption could not hold true. Now $a_1 \cong_\nu a_2$, and thus, by applying Proposition 4.13 to each monomial in F, we can conclude that the same monomial dominates also at a_2. Since this monomial is ghost-valued, we have $F(a_2) \in R_{>1}$. If there are more than one monomial dominating at a_1, then by denoting two of these dominating monomials as F_i and F_j, we have $F(a_1) \cong_\nu F_i(a_1) \cong_\nu F_j(a_1)$. Since $a_1 \cong_\nu a_2$, Proposition 4.13 implies $F(a_2) \cong_\nu F_i(a_2) \cong_\nu F_j(a_2)$. Hence there are at least two monomials dominating at a_2, so it must be $F(a_2) \in R_{>1}$.

In the above reasoning, we can change the roles of a_1 and a_2, and thus, it also holds that $F(a_2) \in R_{>1}$ implies $F(a_1) \in R_{>1}$. Therefore $F(a_1) \notin R_{>1}$ implies $F(a_2) \notin R_{>1}$.

Clearly, the roots following Definition 6.40 satisfy (FP-A). Since the aforementioned three sets correspond to each other at tangible values, and since both (FP-C) and (FP-A) hold, we achieve a full correspondence (at any values) between all these three sets by using the definition proposed.

Cluster locus is the complement of corner locus in respect to total locus. If we define corner roots in the same way as earlier, but change the definition of total locus to follow Definition 6.40, then cluster locus changes, too. A new definition for a cluster root would be as follows: A point a (of a semiring R) is a cluster root of a polynomial F (over R), if F has a single monomial dominating at a such that the coefficient of the monomial is ghost. In this way, we have not the undesired situation that ghost elements are most often roots.

However, Definition 6.40 has the drawback that it misses the property, according to which the value of a polynomial is ghost exactly at those points that are the roots of the polynomial. Or otherwise we need a third kind of root set.

6.5 Coordinate semiring

The fact that a congruence variety is an algebraic set can be applied in the context of coordinate semirings. As in usual algebraic geometry, a (tropical) coordinate semiring is determined based on an affine algebraic set. However, Lemma 6.29 enables us to determine a coordinate semiring based on a congruence variety.

We start by defining a coordinate semiring according to [18, p. 31] and [21, p. 10], as follows.

Definition 6.41. Let K be a 1-semifield and $X \subset K^n$ an affine algebraic set. Consider the restriction map

$$\begin{aligned} \mathrm{Pol}(K^n, K) &\to \mathrm{Fun}(X, K) \\ f &\mapsto f|_X. \end{aligned}$$

The *coordinate semiring* of X is the image of the above map, and it is denoted as $K\lfloor X \rfloor$.

Remark. Actually, $K\lfloor X \rfloor$ is the set of polynomial functions from X to K, which is the same as the set $\mathrm{Pol}(X, K)$. These sets are the same, if we think a function as a set of pairs, the first component of which is an element of the domain set and the second one of which is an element of the target set.

Note that the denotation $K\lfloor X \rfloor$ corresponds to that introduced in Definition 4.10.

The following proposition describes the connection between coordinate semirings and congruences.

Proposition 6.42. *Let K be a 1-semifield with at leat two layers, $S \subset K^n$ a subset, and $X \subset S$ a congruence variety. Then*

$$K\lfloor X \rfloor \cong \mathrm{Pol}(S, K)/\Omega_X.$$

Proof. Since K has at least two layers, Lemma 6.29 implies that X is an algebraic set. Therefore we can write $K\lfloor X \rfloor$ to denote a coordinate semiring†, as introduced in Definition 6.41.

Consider the composition map

$$\varphi : \mathrm{Pol}(S, K) \xrightarrow{\psi} \mathrm{Pol}(X, K) \xrightarrow{\pi} K\lfloor X \rfloor.$$

Here ψ is a surjective homomorphism, since $X \subset S$. Namely, ψ maps each polynomial function to its restriction to X. Furthermore, π is an identity map, as discussed in the remark after Definition 6.41. Since both the homomorphisms are surjective, the whole composition map is a surjective homomorphism. Moreover,

$$\begin{aligned}
\mathrm{Ker}\,\varphi &= \{(f, g) \in \mathrm{Pol}(S, K)^2 \mid \pi(\psi(f)) = \pi(\psi(g))\} \\
&= \{(f, g) \in \mathrm{Pol}(S, K)^2 \mid f|_X = g|_X\} \\
&= \{(f, g) \in \mathrm{Pol}(S, K)^2 \mid f(a) = g(a) \text{ for all } a \in X\} \\
&= \Omega_X,
\end{aligned}$$

when the claim follows from the isomorphism law, given in Proposition 5.26. □

Remark. Now X is a congruence variety, and thus, $X = V(\Omega_X)$. Therefore the claim can be written as

$$K\lfloor V(\Omega_X) \rfloor \cong \mathrm{Pol}(S, K)/\Omega_X.$$

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